Space-Variant Computer Vision: A Graph Theoretic Approach

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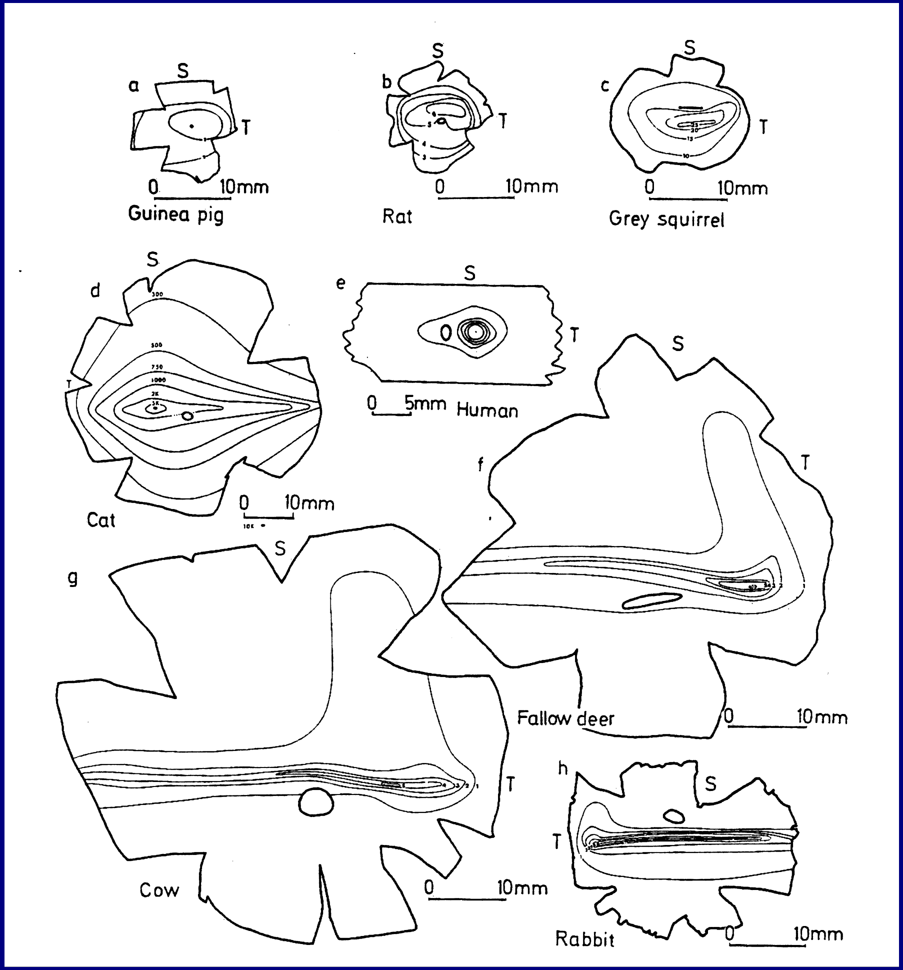
Outline of talk

- Space-variant vision - Why and how of graph theory
- Anisotropic interpolation
- Isoperimetric segmentation
- Topology and numerical efficiency
- Multiresolution segmentation
- Conclusion - Future work

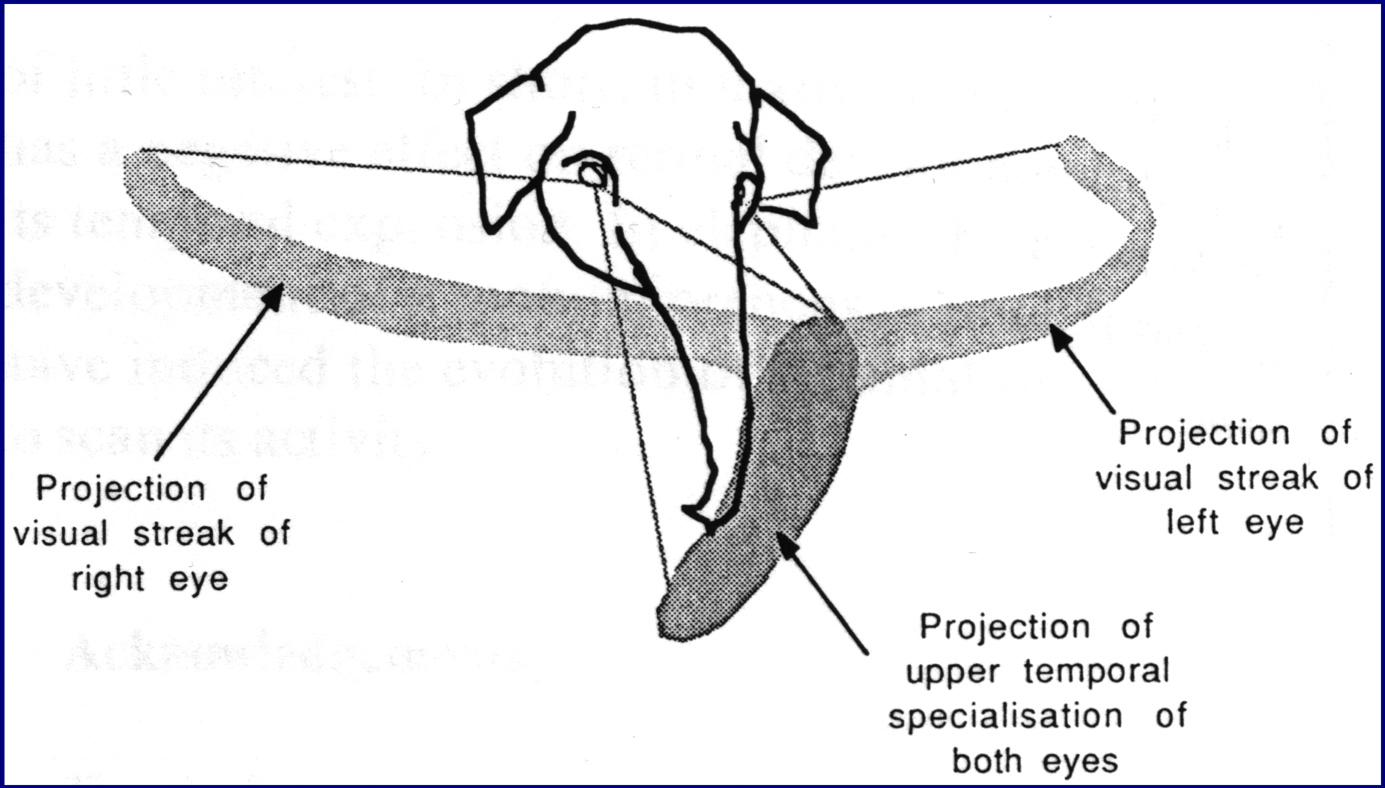
Biological visual sampling

Why has evolution driven this process?
(Hughes, 1977)



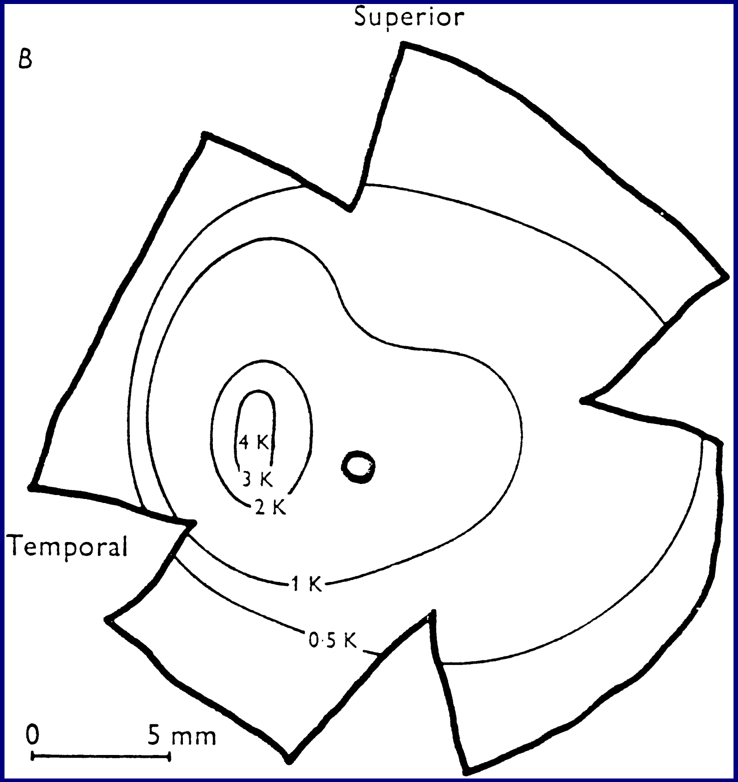
Biological visual sampling

Visual archetecture satisfies the needs of the system

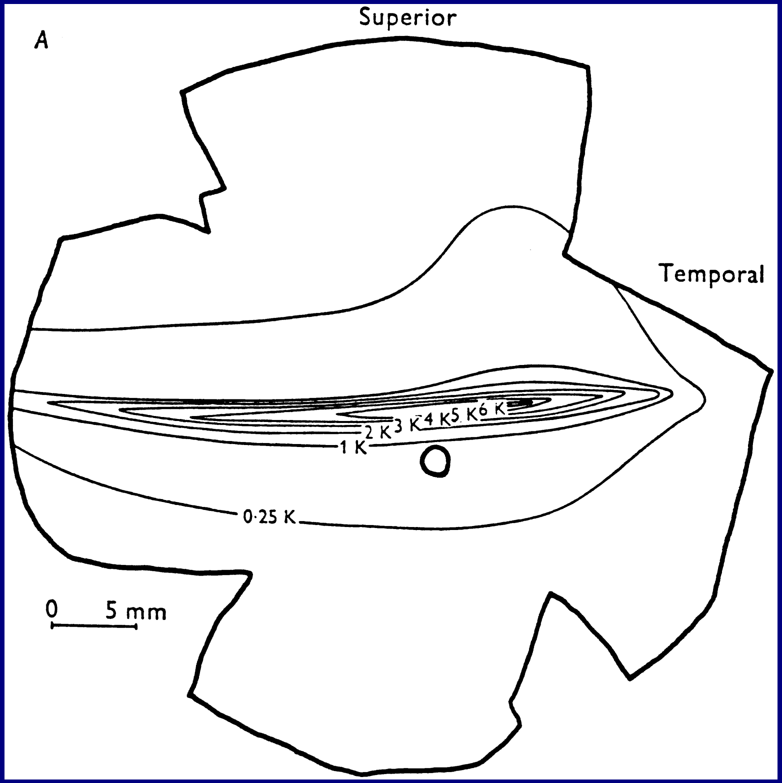


Biological visual sampling

The "Terrain Theory" of Hughes (1977)



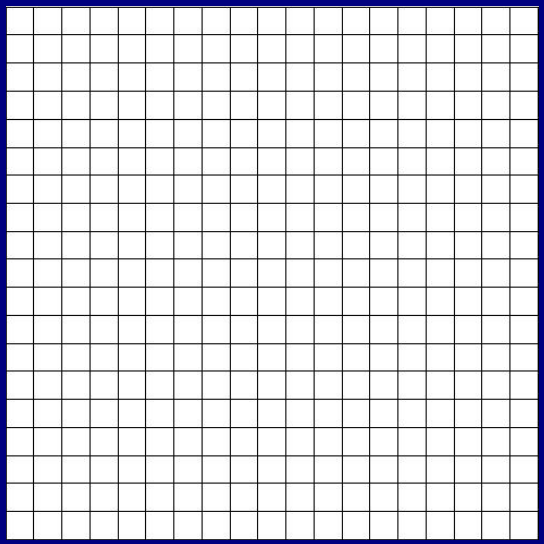
(a) Tree Kangaroo



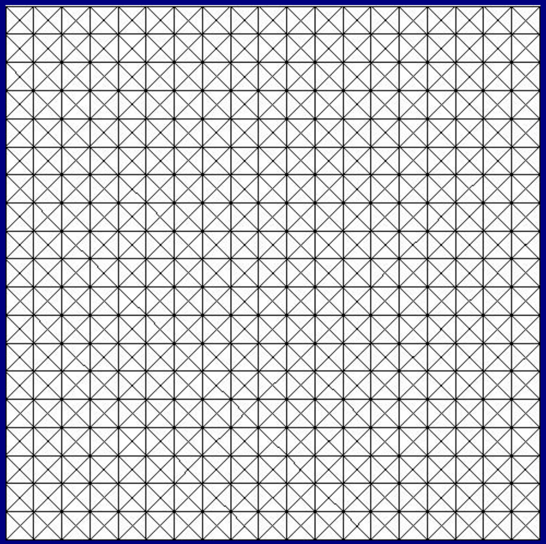
(b) Ground Kangaroo

Biological visual sampling

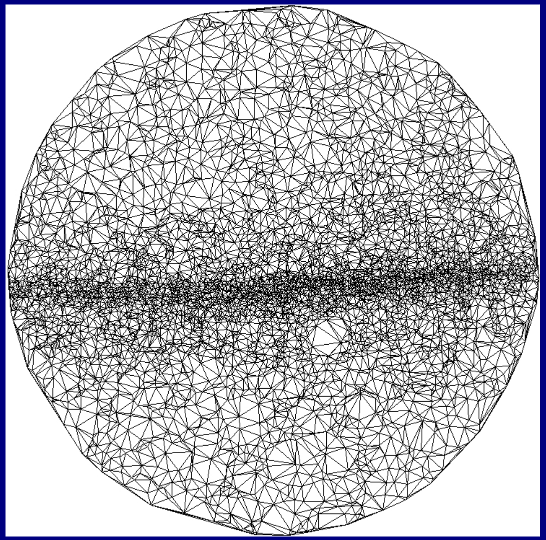
Problem: How to apply computer vision techniques to space-variant images?



(a) 4-Connected



(b) 8-Connected



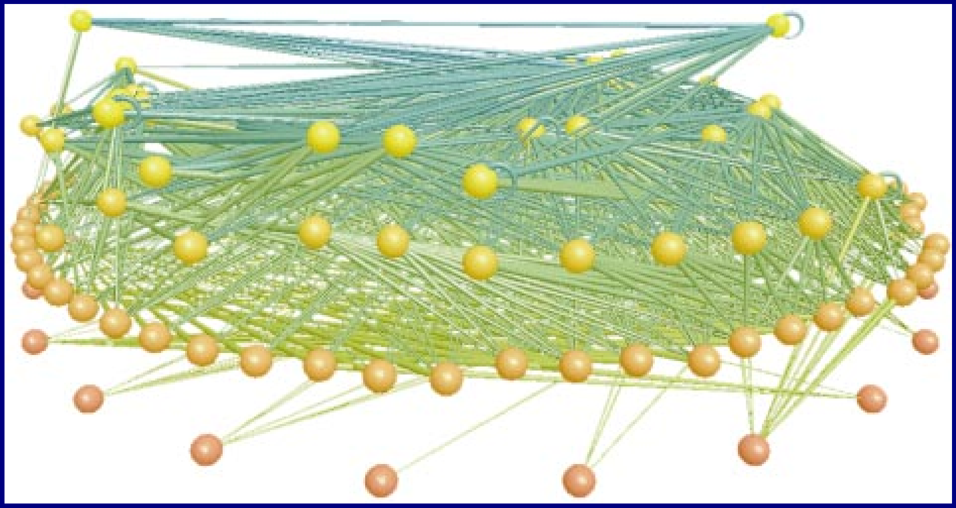
(c) Kangaroo

Biological visual sampling

Approach: Graph theory

Nodes correspond to pixels and node density corresponds to resolution

- Naturally extends to 3D or data clustering
- Applies to any field defined on nodes (e.g., image intensity, coordinates)



Advantages of graph theory

Why should we be interested in graph-theoretic approaches to vision?

- Recent success using graph theory in computer vision
- Global-local interactions
- Dimension independent
- Analogies between graphs, matrices, vector calculus and circuit theory allows transfer of ideas and intuition

Graph theory

How to use concepts from calculus on a graph?

A **graph** is a pair $G = (V, E)$ with vertices (nodes) $v \in V$ and edges $e \in E \subseteq V \times V$. An edge, e, spanning two vertices, v_i and v_j, is denoted by e_{ij}. A **weighted graph** has a value (typically nonnegative and real) assigned to each edge, e_{ij}, denoted by $w(e_{ij})$ or w_{ij}.

$$A_{e_{ij}v_k} = \begin{cases} +1 & \text{if } i = k, \\ -1 & \text{if } j = k, \\ 0 & \text{otherwise.} \end{cases}$$

$$L_{v_iv_j} = \begin{cases} d_i & \text{if } i = j, \\ -w(e_{ij}) & \text{if } e_{ij} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

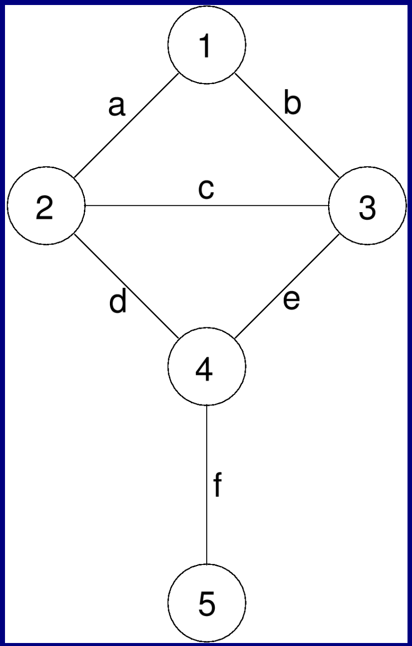
$$C_{e_{ij}e_{ks}} = \begin{cases} w(e_{ij}) & \text{if } i = k, j = s, \\ 0 & \text{otherwise.} \end{cases}$$

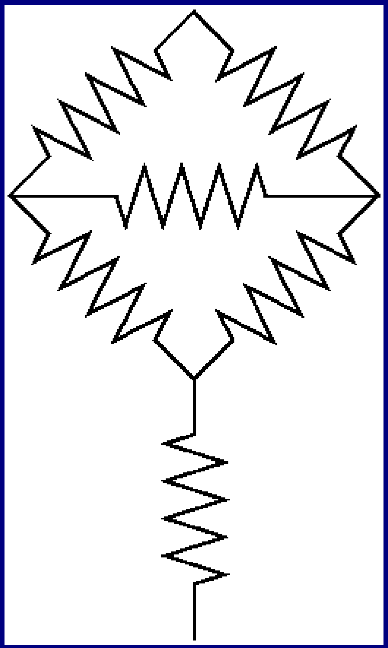
$$d_i = \sum_{e_{ij}} w(e_{ij}) \;\; \forall \; e_{ij} \in E$$

$$L = A^T C A$$

Graph theory - Circuit analogy

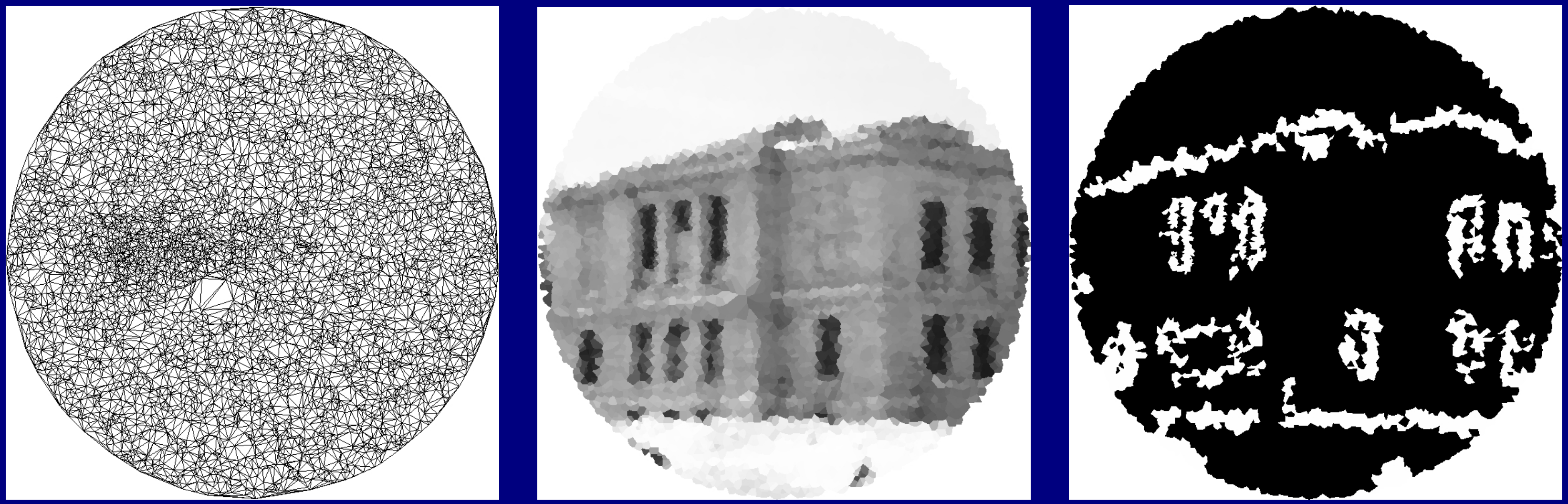
$A^T y = f$	Kirchhoff's Current Law
$Cp = y$	Ohm's Law
$p = Ax$	Kirchhoff's Voltage Law





Graph theory

Allows straightforward translation of computer vision concepts



(a) Structure (b) Image (c) Edge-detection

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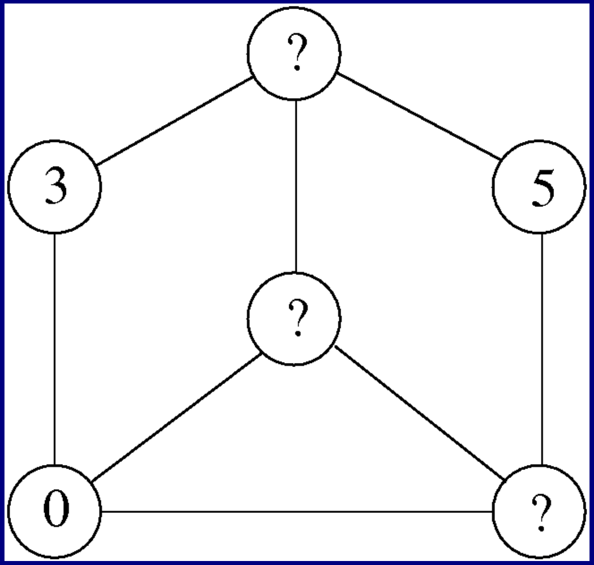
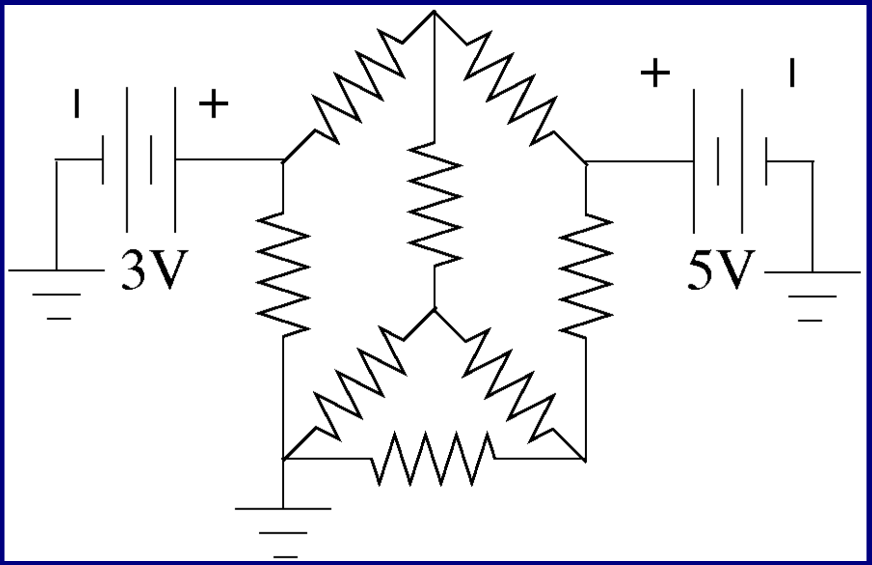
Graph interpolation

Problem: How to interpolate unknown nodal values between known nodal values?

Answer: Solve the combinatorial Dirichlet Problem using Dirichlet boundary conditions at the known values



(a) Graph



(b) Circuit

The Dirichlet Problem

Continuous — Dirichlet integral

$$D[u] = \frac{1}{2}\int_\Omega |\nabla u|^2 dV$$

Laplace equation

$$\nabla^2 u = 0$$

Combinatorial — Dirichlet integral

$$D[x] = \frac{1}{2}x^T A^T C A x$$

Laplace equation

$$A^T C A x = L x = 0$$

Dirichlet Problem

How are the boundary conditions actually incorporated?

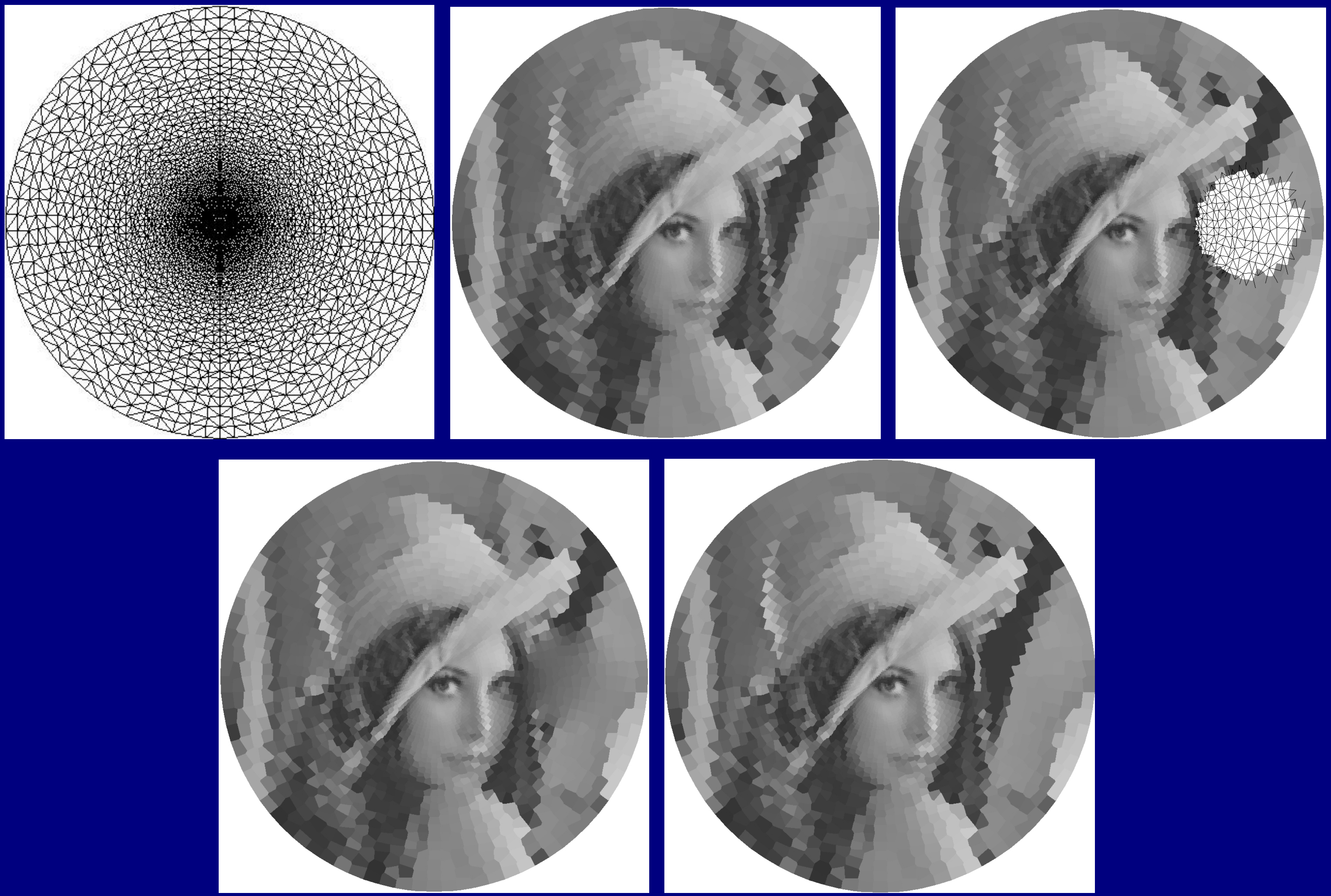
$$D[x] = \frac{1}{2} x^T L x$$

$$D[x_i] = \frac{1}{2} \left[x_b^T x_i^T \right] \begin{bmatrix} L_b & R \\ R^T & L_i \end{bmatrix} \begin{bmatrix} x_b \\ x_i \end{bmatrix}$$

$$= x_b^T L_b x_b + 2 x_i^T R^T x_b + x_i^T L_i x_i$$

$$L_i x_i = -R^T x_b$$

Does it work?



Dirichlet Problem - So what?

Connection with anisotropic diffusion (Perona and Malik, 1990) suggests use for early vision processing

Diffusion equation

$$\frac{dx}{dt} = Lx$$

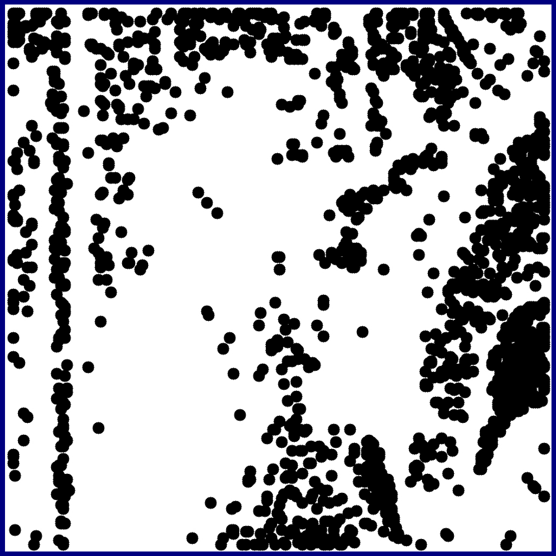
Laplace equation

$$0 = Lx$$

Dirichlet Problem

Anisotropic Laplacian smoothing

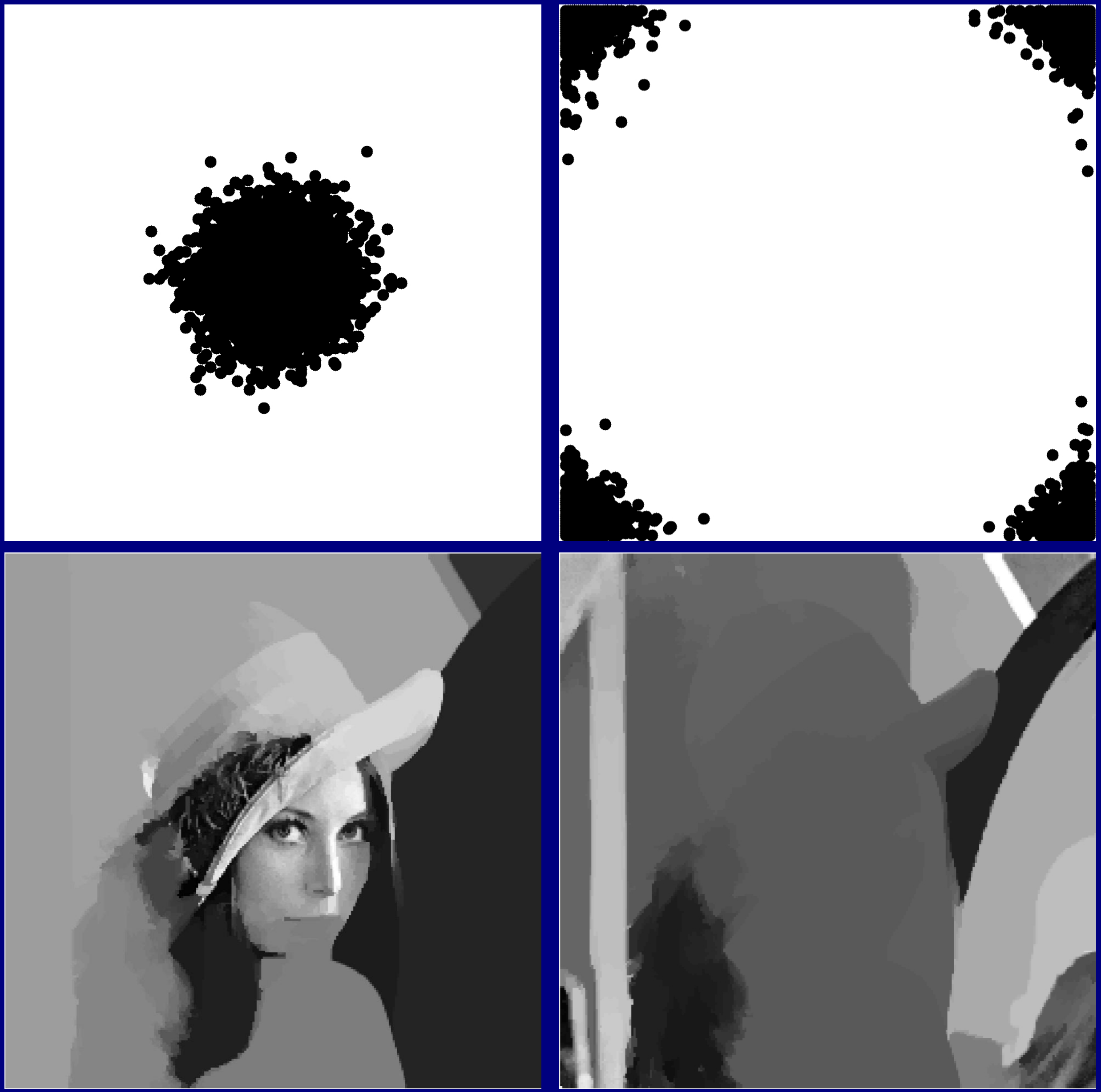
(a) Original



(b) Samples

(c) Interpolation

Space-variant sampling



Laplace vs. Diffusion

Conclusion

- Steady state vs. transient
- Diffusion requires stopping parameter (i.e., time)
- "Sampling" of Laplace allows more or less smoothing in different regions

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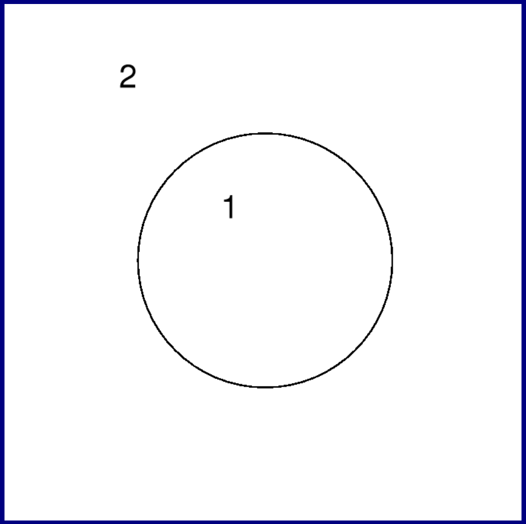
Isoperimetric Problem

Problem: For a given volume, what shape has the smallest perimeter?



Enclosing a volume with a boundary may be considered as a *separation* of the space.

Segmentation: View image as a discrete geometry where the image values specify the metric and find the partitions that satisfy the isoperimetric problem.

Isoperimetric Problem

For an arbitrary space, the problem is much more difficult.

Formally, the **isoperimetric constant** for a space is given as

$$h = \inf_{S} \frac{|\partial S|}{\mathrm{Vol}_S},$$

for any subset of points, S.

Isoperimetric Problem

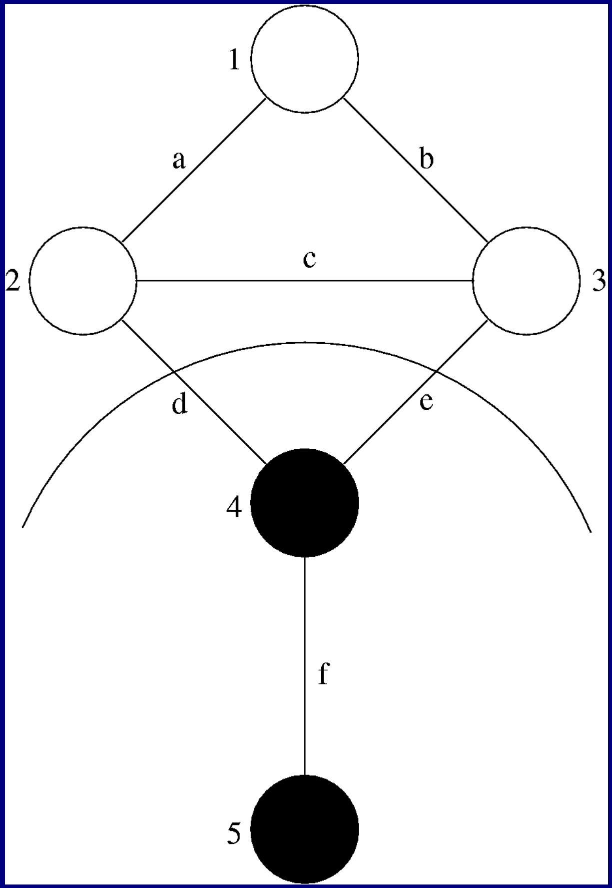
How to define problem for a discrete geometry (graph)?

Instead of points, S is a set of nodes.

$$S = \{4, 5\}$$

$$\overline{S} = \{1, 2, 3\}$$

$$\partial S = \{d, e\}$$



Graph formulation

Define an indicator vector

$$x_i = \begin{cases} 0 & \text{if } v_i \notin S, \\ 1 & \text{if } v_i \in S. \end{cases}$$

Note that specification of x defines a partition.

How to define perimeter?

$$|\partial S| = x^T L x$$

Graph formulation

How to define volume?

Nodal Volume	$\mathrm{Vol}_S = x^T r$
Normalized Volume	$\mathrm{Vol}_S = x^T d$

Where r is the vector of all ones and d is the vector of node **degree.**

Graph formulation

As a measure of partition quality, define the **isoperimetric ratio** as

$$h(x) = \frac{x^T L x}{x^T d}.$$

Strategy: Relax x to take real values and use Lagrange multiplier to perform a constrained minimization of the perimeter with respect to a constant volume. Namely, minimize $x^T L x$ subject to the constraint

$$x^T d = k.$$

Graph formulation

Define the function $Q(x)$ as

$$Q(x) = x^T Lx - \Lambda(x^T d - k).$$

Since L is positive semi-definite (Biggs, 1974), $Q(x)$ takes minima at critical points satisfying

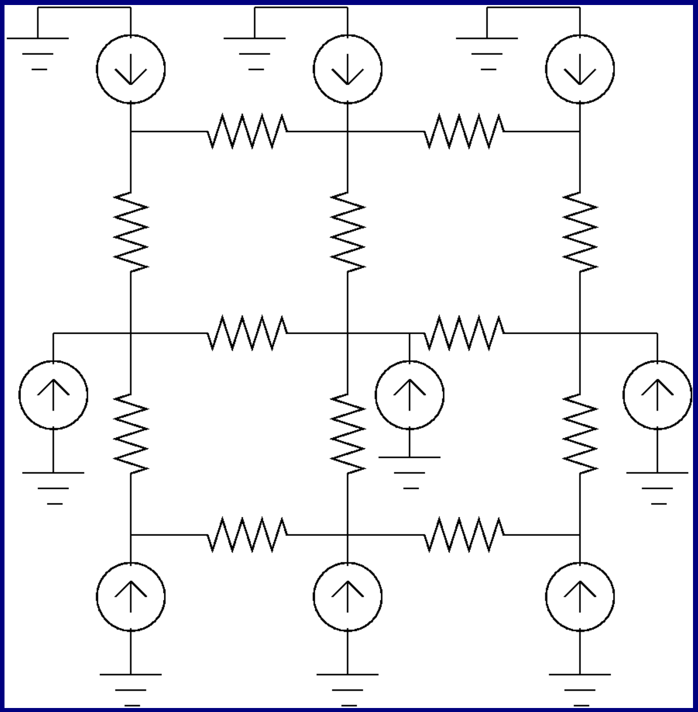
$$2Lx = \Lambda d.$$

Problem: Equation is singular - For a connected graph, L has a nullspace spanned by r.

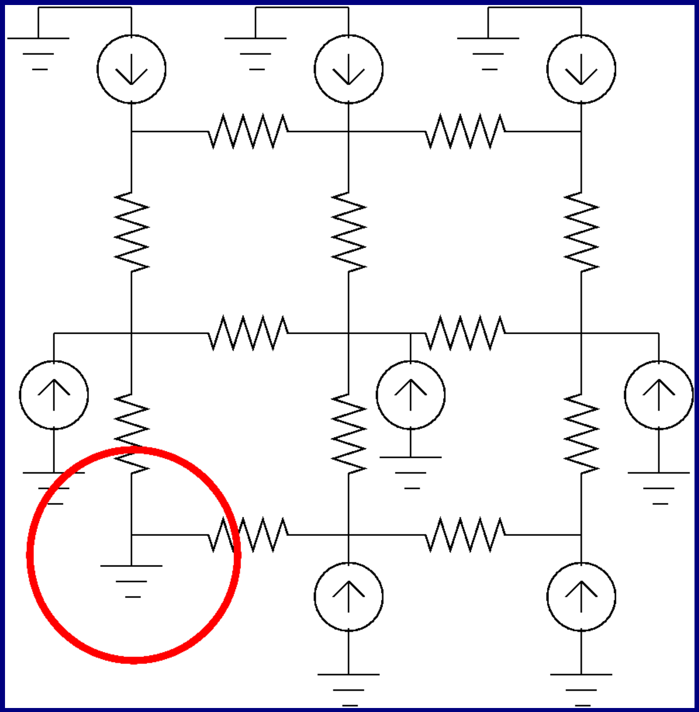
Circuit analogy

Same equation occurs in circuit theory (Branin Jr., 1966) for a resistive network powered by current sources. Must *ground* the circuit in order to find potentials.

$Lx = d$

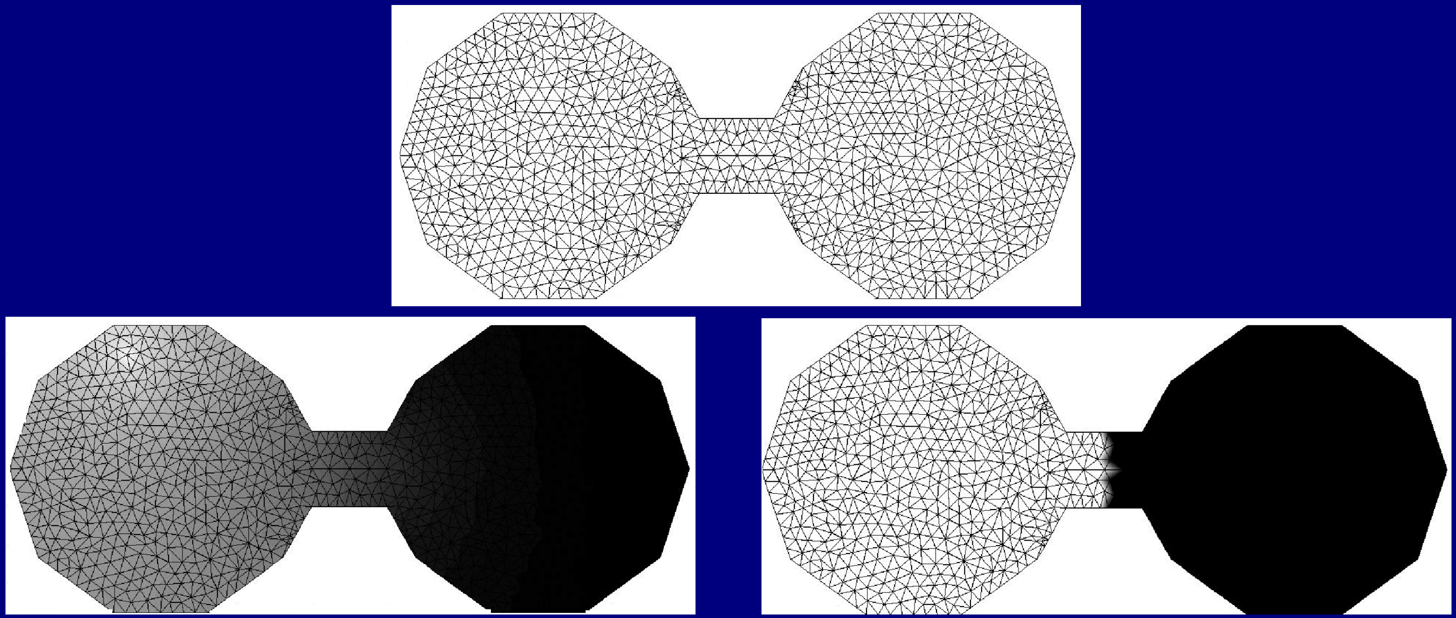


$L_0 x_0 = d_0$



Potentials

When $L_0 x_0 = d_0$ is solved, a set of potentials is assigned to each node that must be thresholded in order to produce a partition.



Lemma: For any threshold, the set of nodes with potentials lower than the threshold must be connected.

Proof: Follows from the mean value theorem with positive sources.

Partitioning algorithm recap

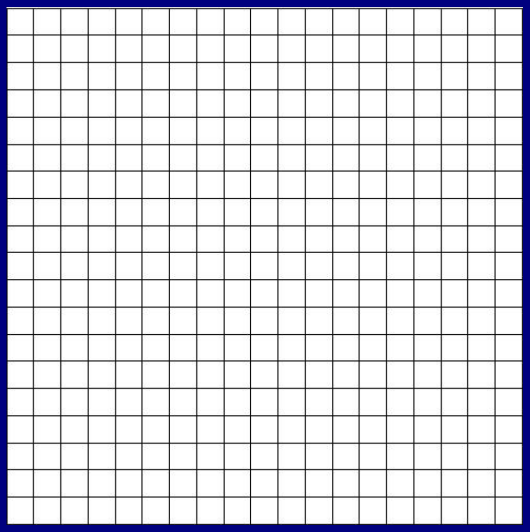
Steps of the partitioning algorithm

1. Generate edge weights based on image/coordinate/sensor data.
2. Choose a ground node, v_g, and form L_0, d_0.
3. Solve $L_0 x_0 = d_0$ for x_0.
4. Choose threshold and divide nodes into the sets with potentials above and below the threshold.

Spectral partitioning

Spectral vs. Isoperimetric

- Eigenvector problem vs. system of linear equations
- If eigenvalue algebraic multiplicity is greater than unity: Any vector in the subspace spanned by the eigenvectors is a valid solution.



Data clustering

The Gestalt clustering challenges of Zahn

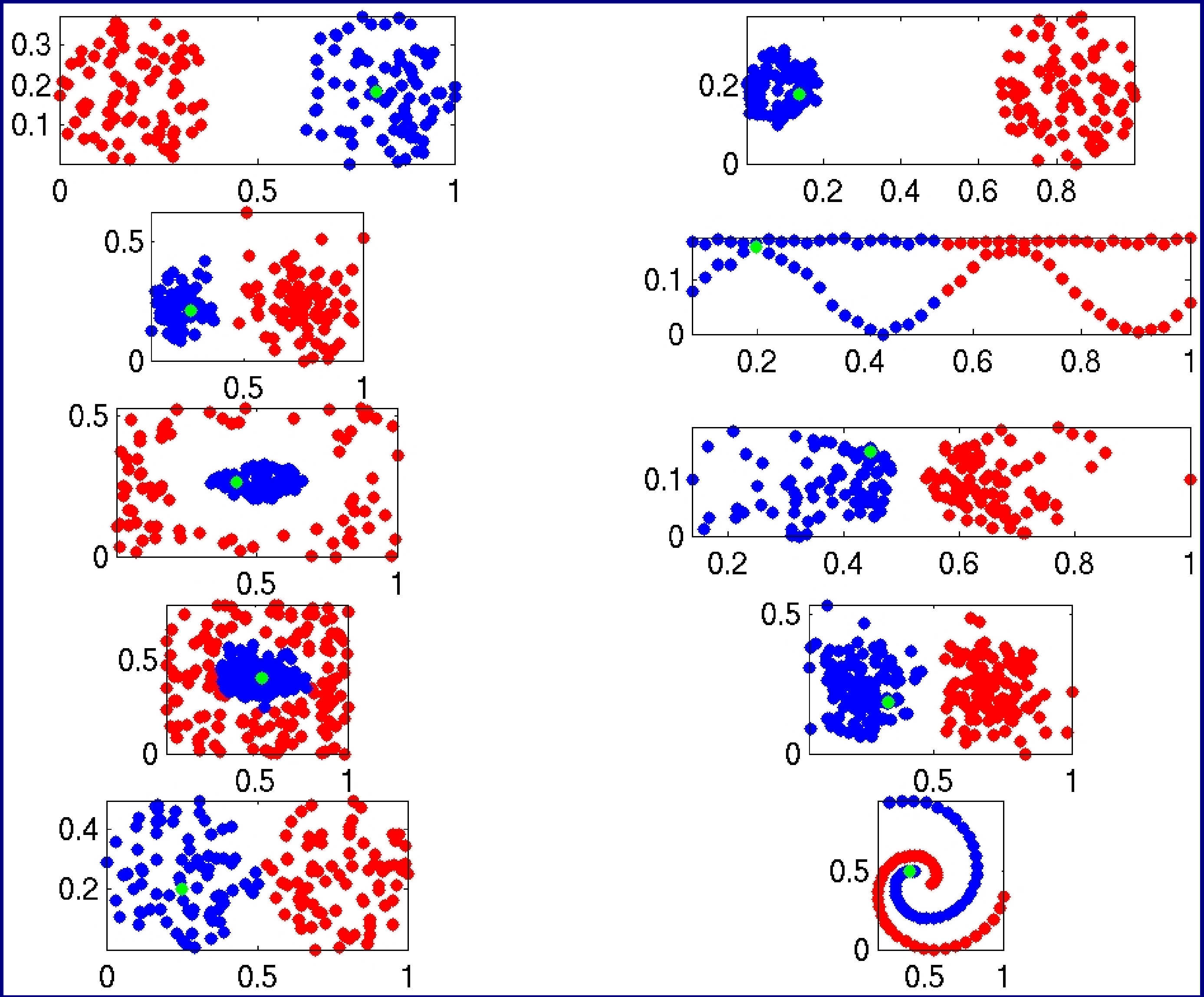


Image processing

Why should this work well?

Image | Potentials | Segmentation

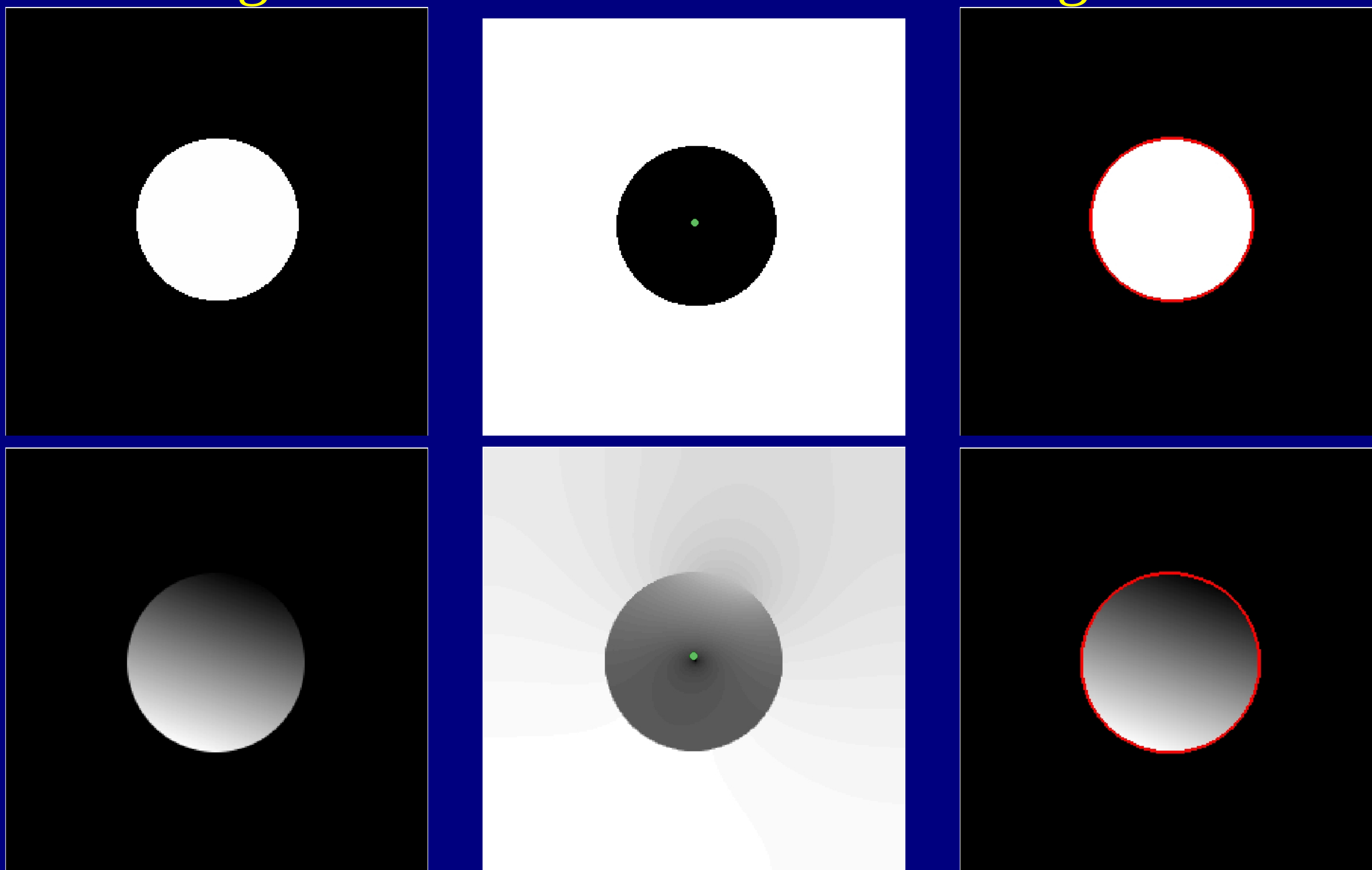


Image processing

What is the effect of choosing ground?

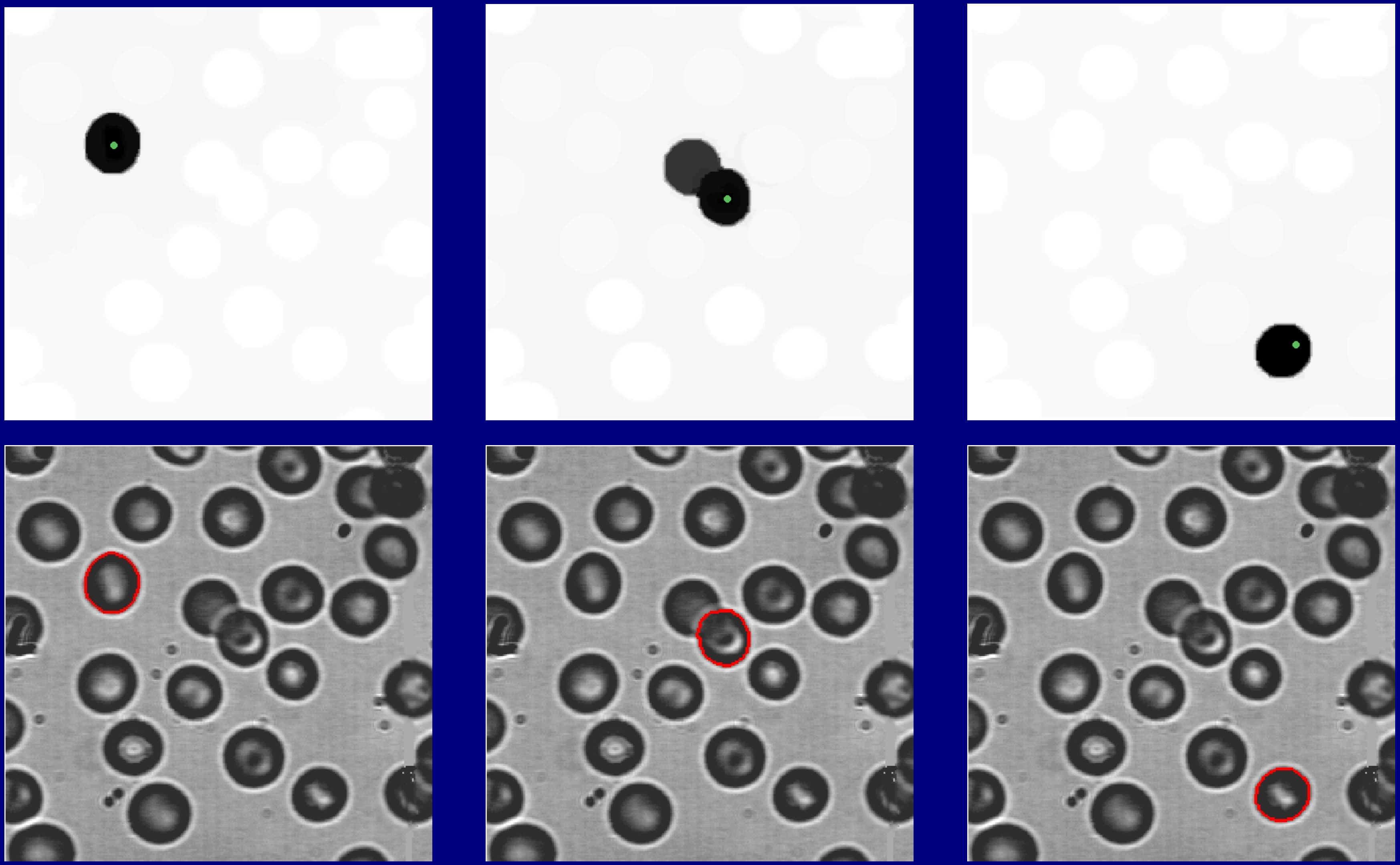


Image processing

How to choose weights?

$$w_{ij} = \exp\left(-\beta |I_i - I_j|\right)$$

We term β the **scale**

(a) $\beta = 30$

(b) $\beta = 50$

Recursive bipartitioning

How to apply isoperimetric partitioning to (unsupervised) image segmentation?

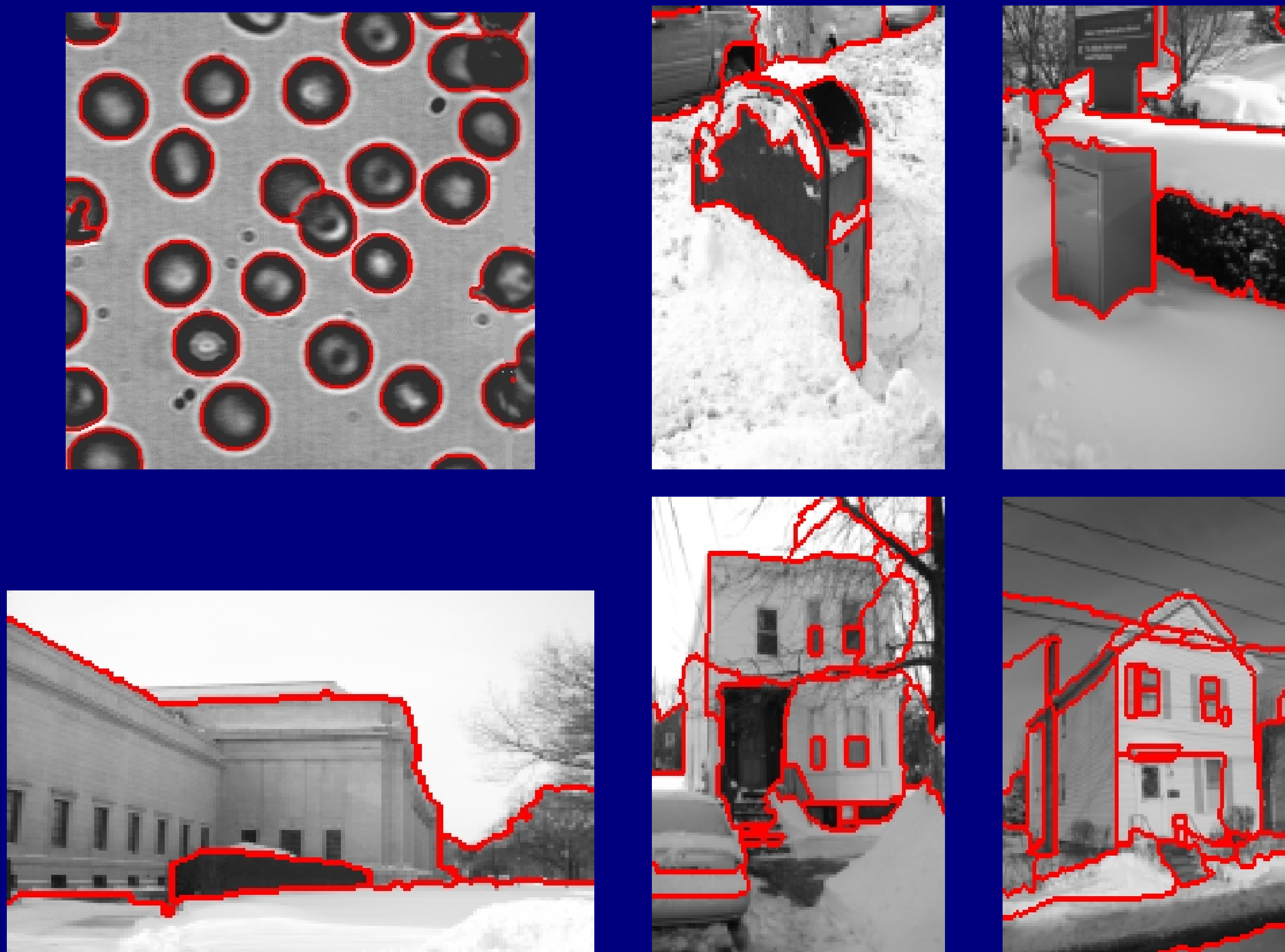
Recursively apply bipartitioning until the isoperimetric ratio of the cut (i.e., the partition quality) fails to satisfy a pre-specified isoperimetric ratio.

Recursive bipartitioning

No user interaction - How to automatically choose ground?

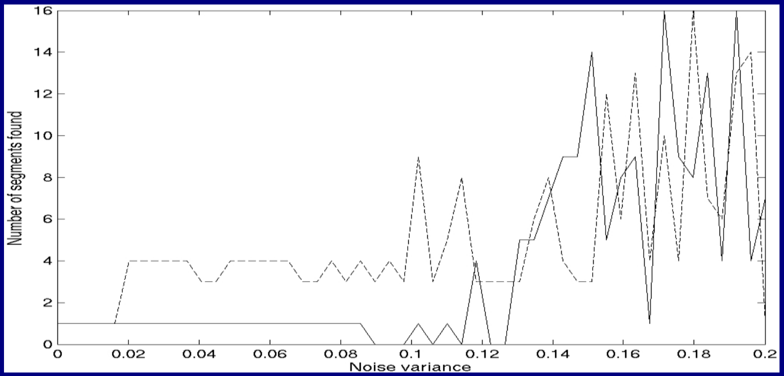
Anderson and Morley (1971) proved that the spectral radius of L, $\rho(L)$, satisfies $\rho(L) \leq 2d_{\max}$. Therefore, grounding the node of highest degree may have the most beneficial affect on the numerical solution to $L_0 x_0 = d_0$.

Recursive bipartitioning

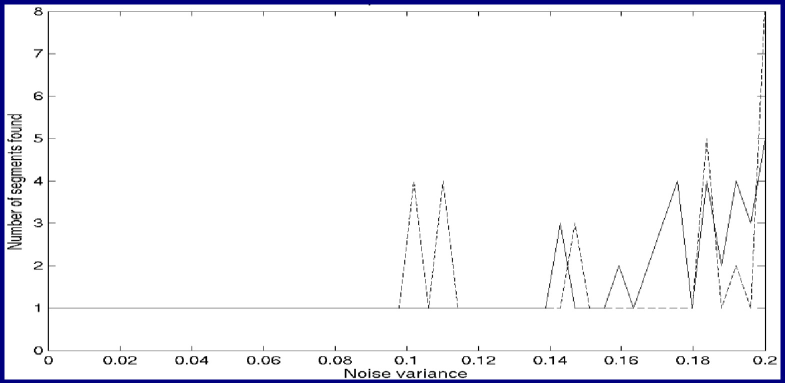


Noise analysis

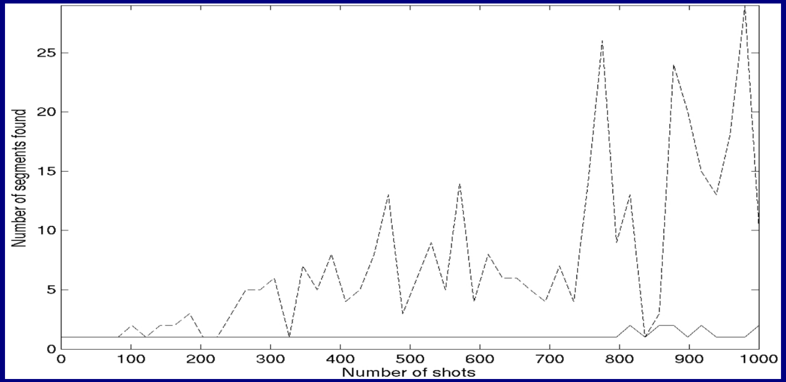
Segmentation of white circle on black background



(a) Additive

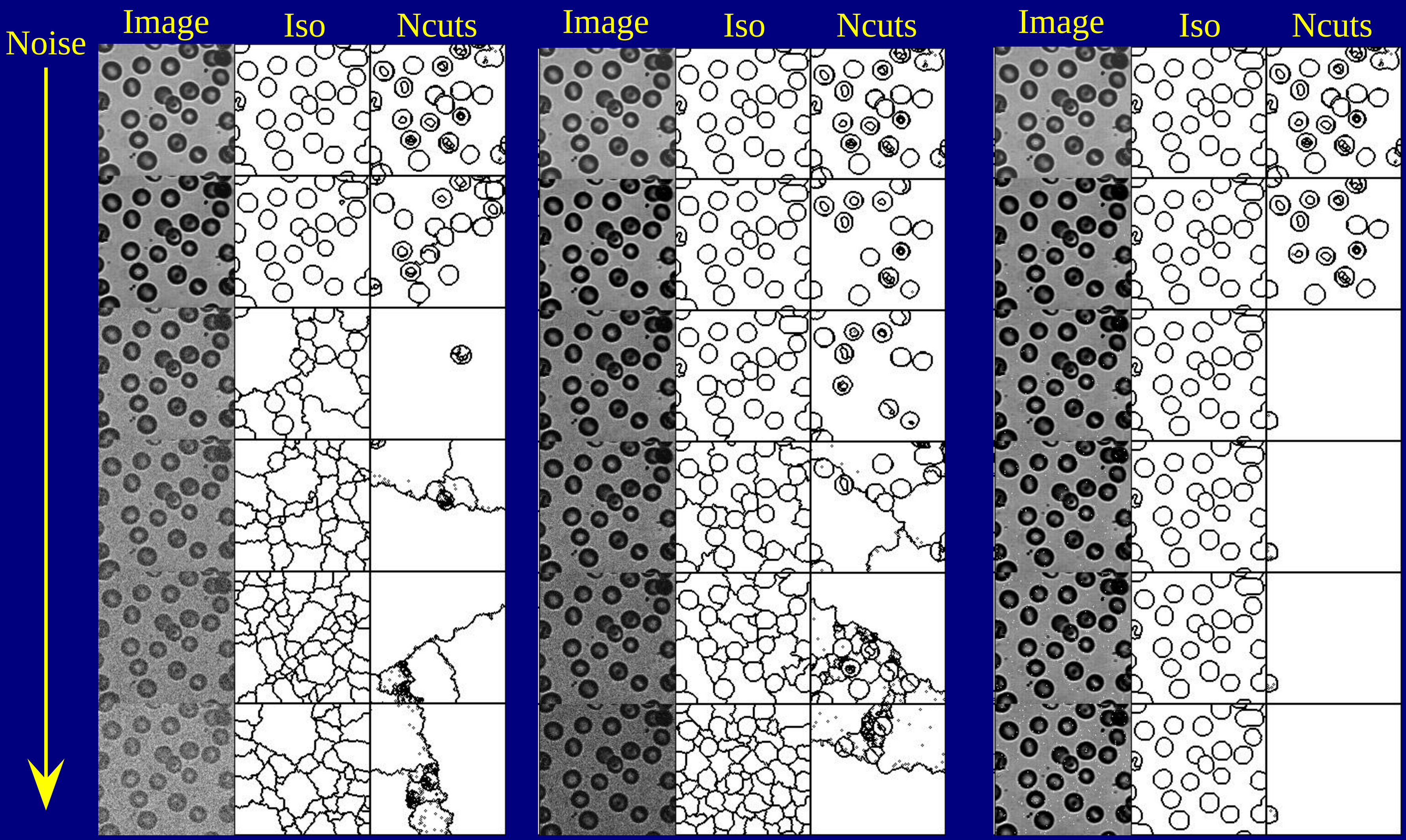


(b) Multiplicative



(c) Shot

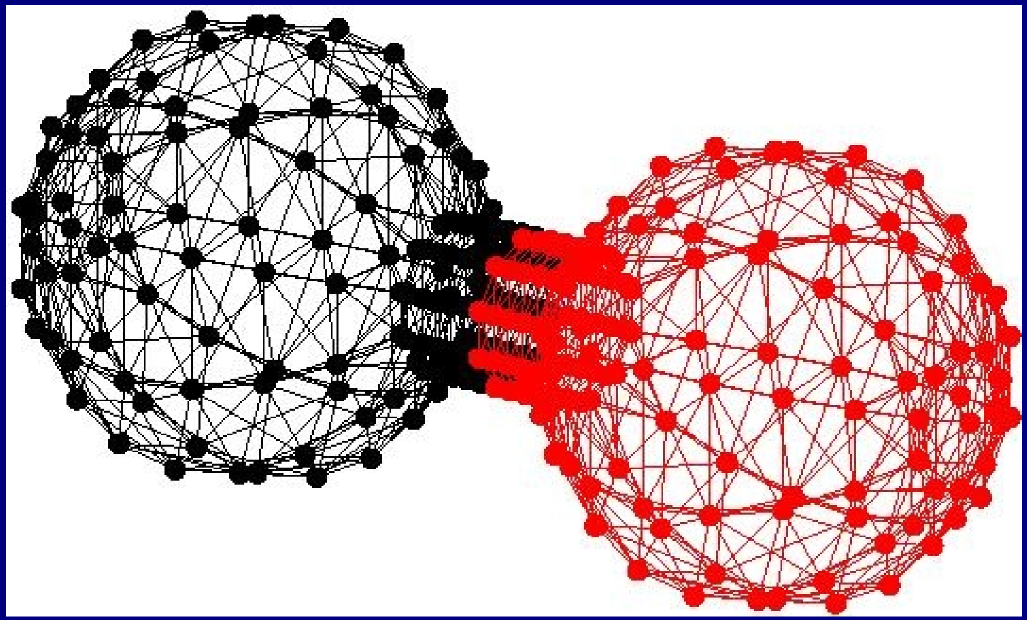
Noise analysis



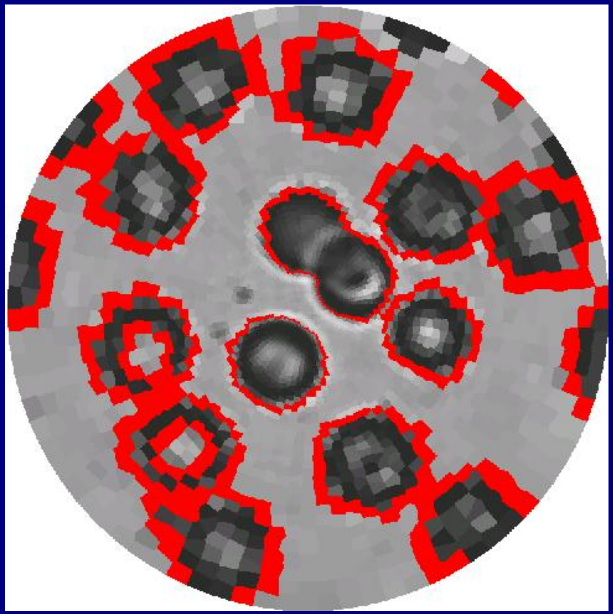
(a) Additive (b) Multiplicative (c) Shot

Other graphs

Because of the flexibility of graph theory, the isoperimetric segmentation algorithm applies to general graphs.



(a) 3D graph



(b) Space-variant

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Krylov subspaces

Krylov subspace methods are the methods of choice for solving both the system of linear equations (conjugate gradients) and the eigenvector problem (Lanczos method).

$$K(A; x_0; k) = \text{span}(x_0, Ax_0, A^2x_0, \ldots, A^{(k-1)}x_0)$$

The solution found at each iteration, i, of conjugate gradients is the solution to $Ax = b$ projected onto the Krylov subspace $K(A, Ax_0 - b, i)$ (Dongarra et al., 1991).

Krylov subspaces

$$K(A; x_0; k) = \text{span}(x_0, Ax_0, A^2 x_0, \ldots, A^{(k-1)} x_0)$$

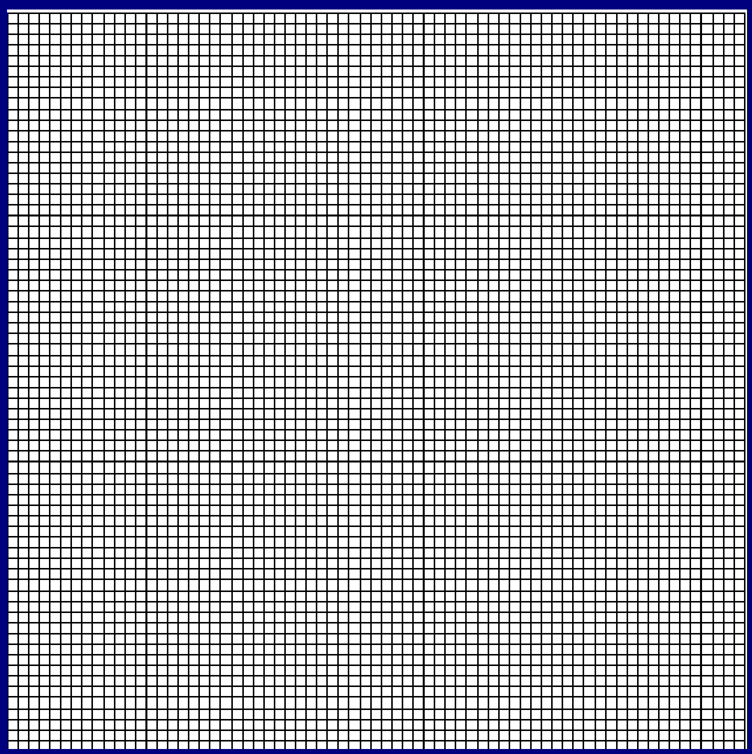
Connection with diffusion

$$x_{i+1} = x_i + \Delta t L x_i$$

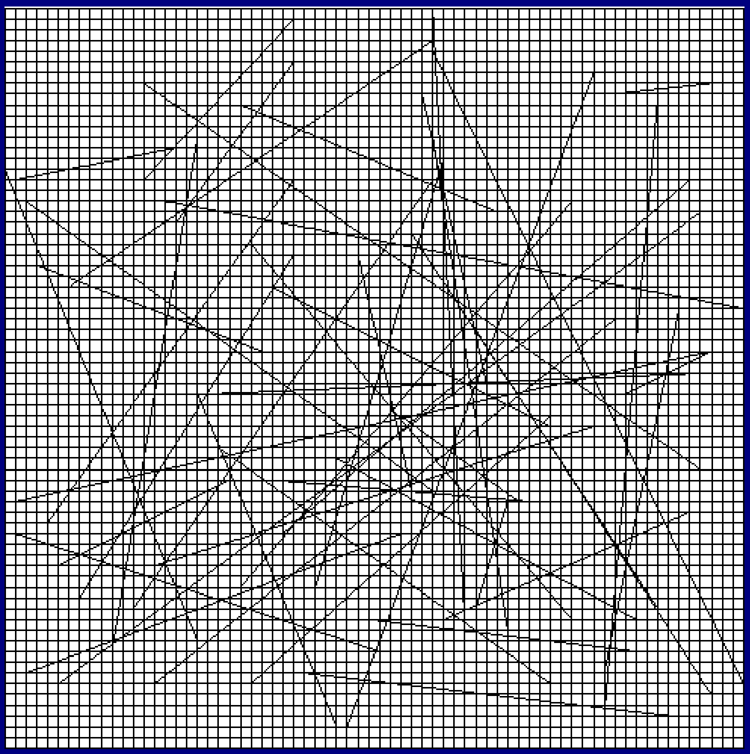
Convergence is directly related to graph diameter.

Krylov subspaces

Idea: Use "small world" graphs to dramatically lower graph diameter with the addition of a minimal number of edges.



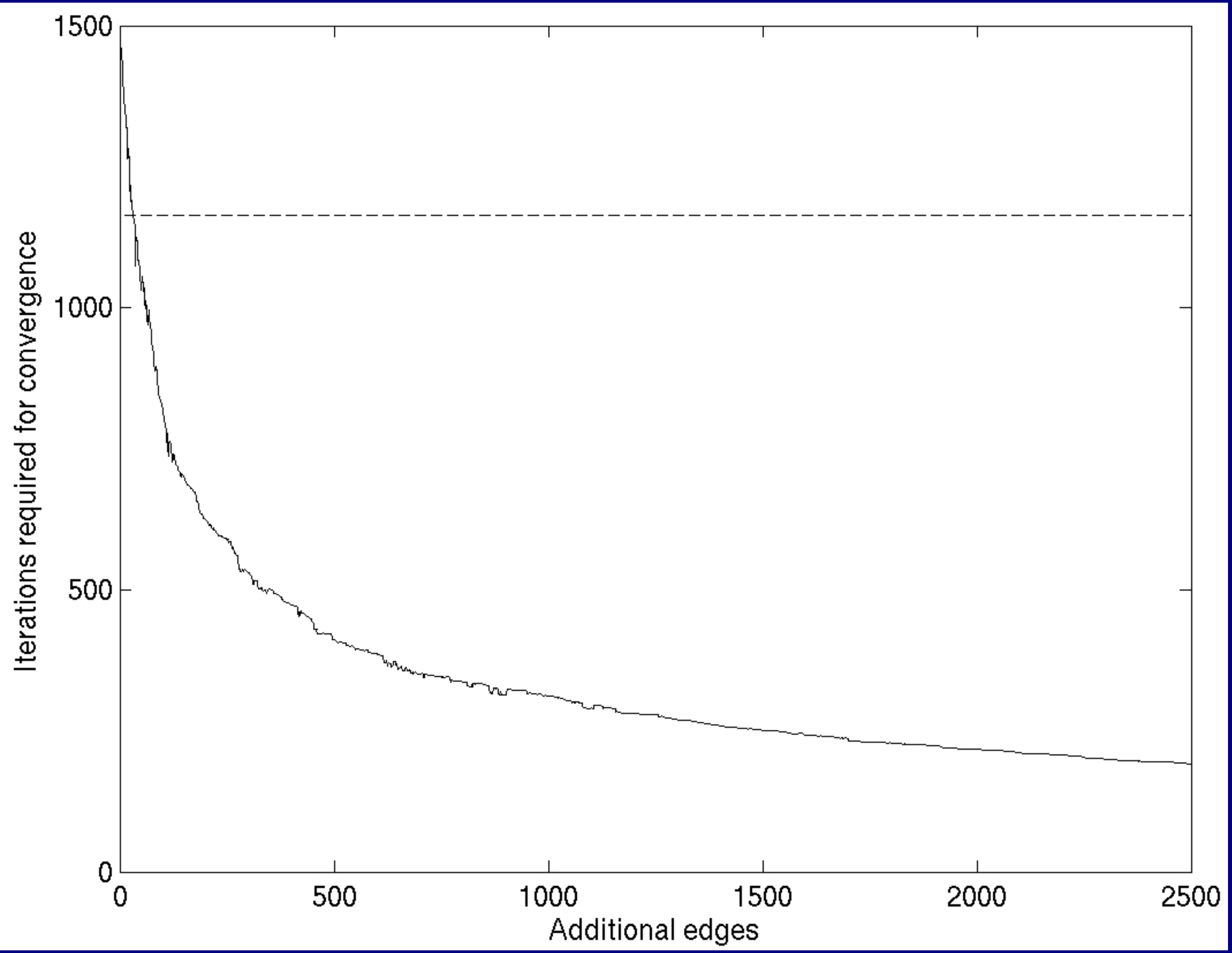
(a) Lattice



(b) Small world

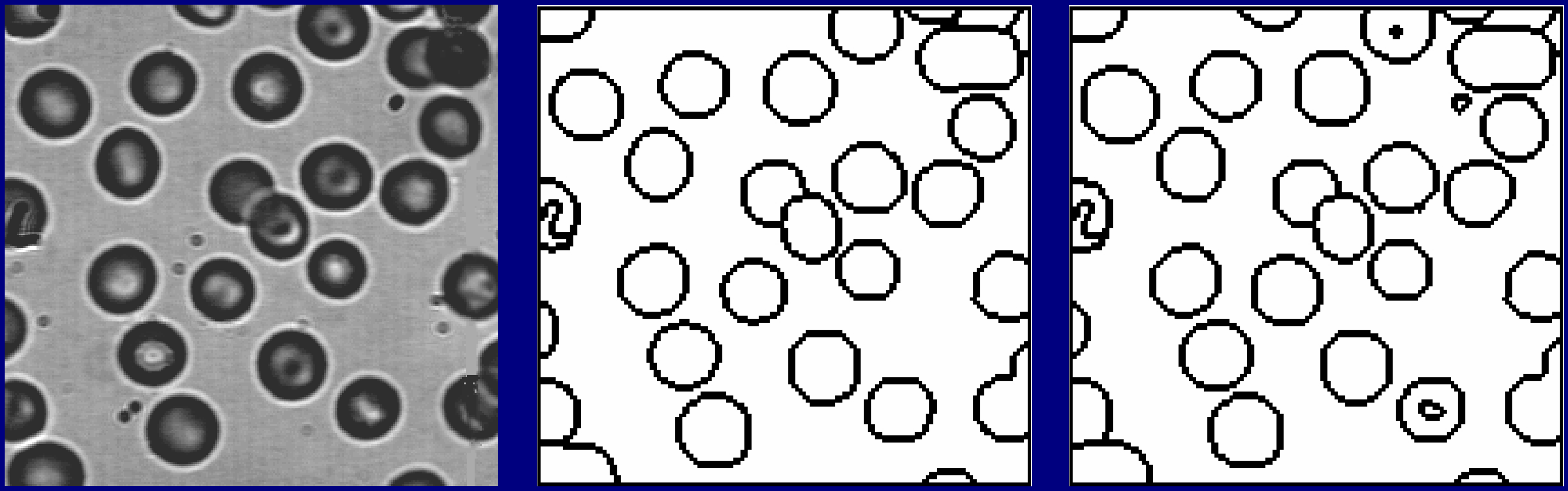
Krylov subspaces

Convergence with new edges added



Krylov subspaces

Results are largely unaffected.



(a) Original (b) Lattice (c) Small World

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Multiresolution analysis

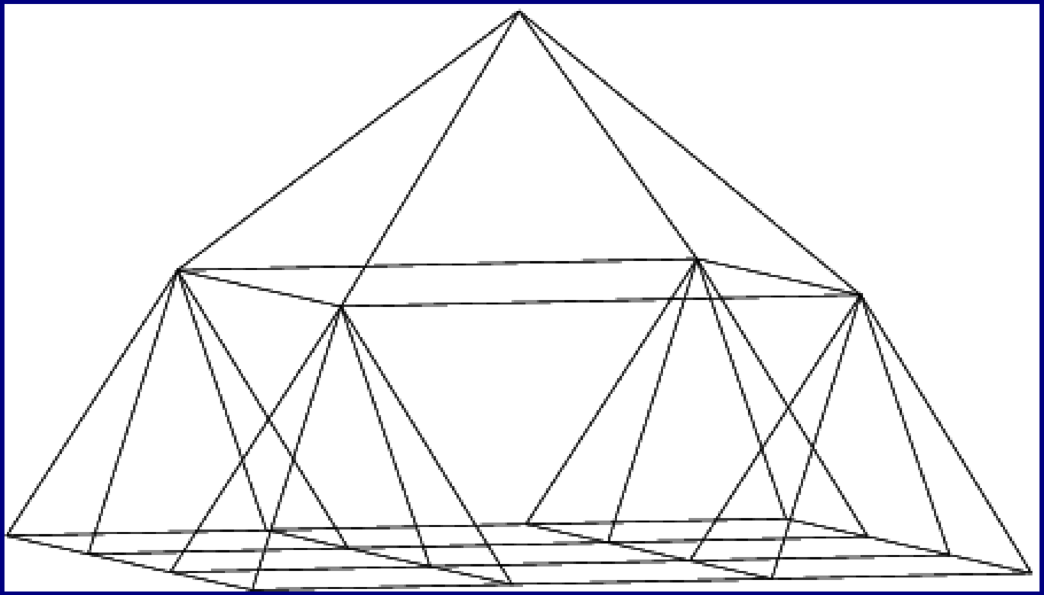
Common approach:

Image $\longrightarrow$ Filter $\longrightarrow$ Process $\longrightarrow$ Backproject solution

Used for speed and resilience to noise

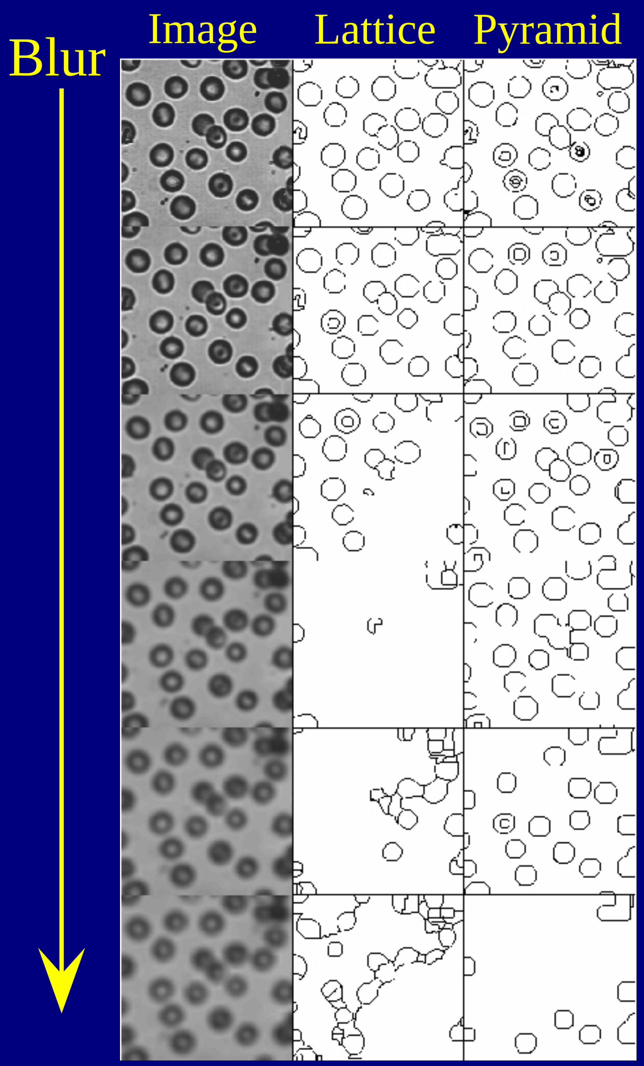
Multiresolution analysis

Idea: Apply a graph-based analysis algorithm to the whole pyramid *as a separate graph*.

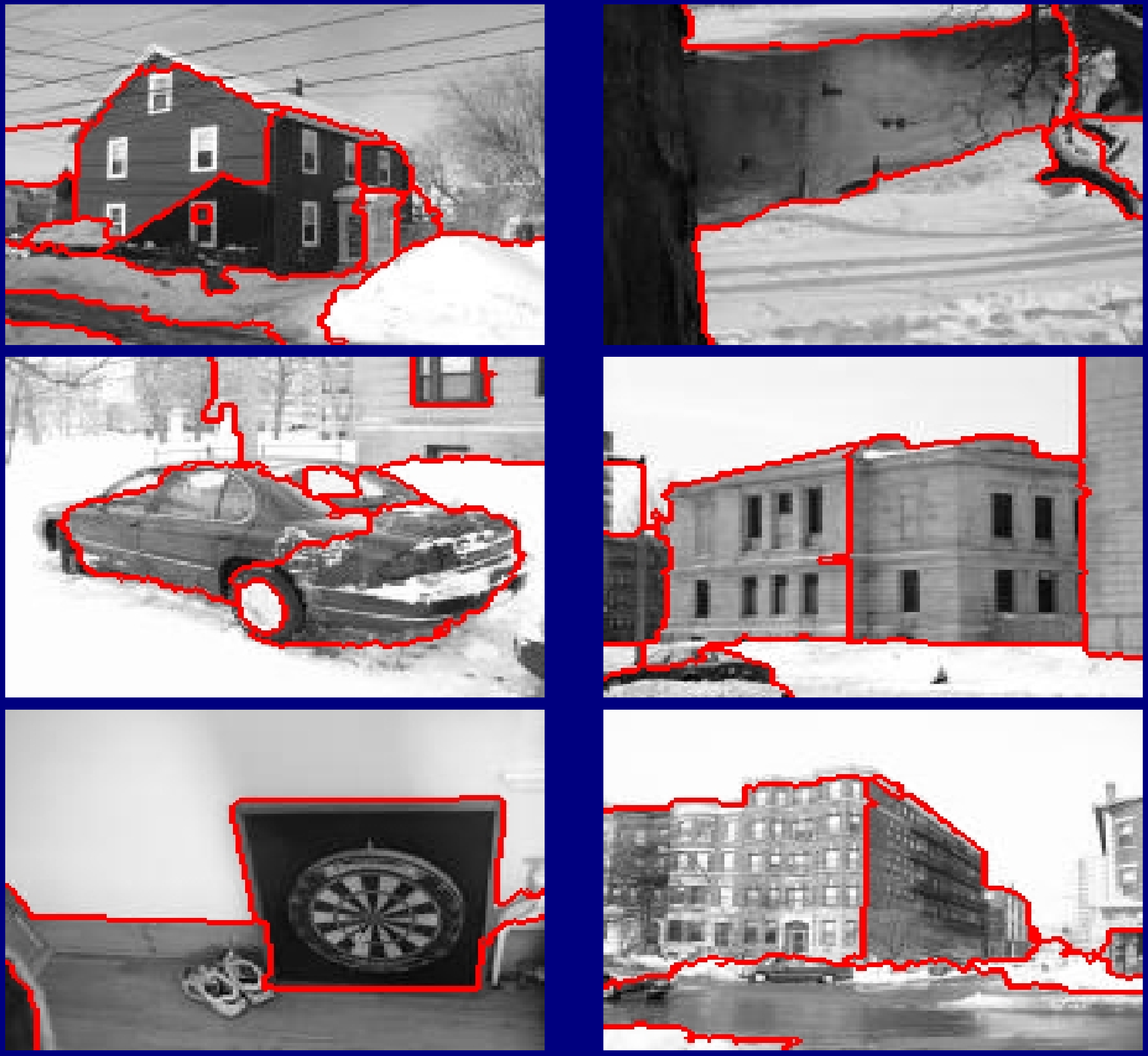


Multiresolution analysis

Long range connections improve performance on blurry images



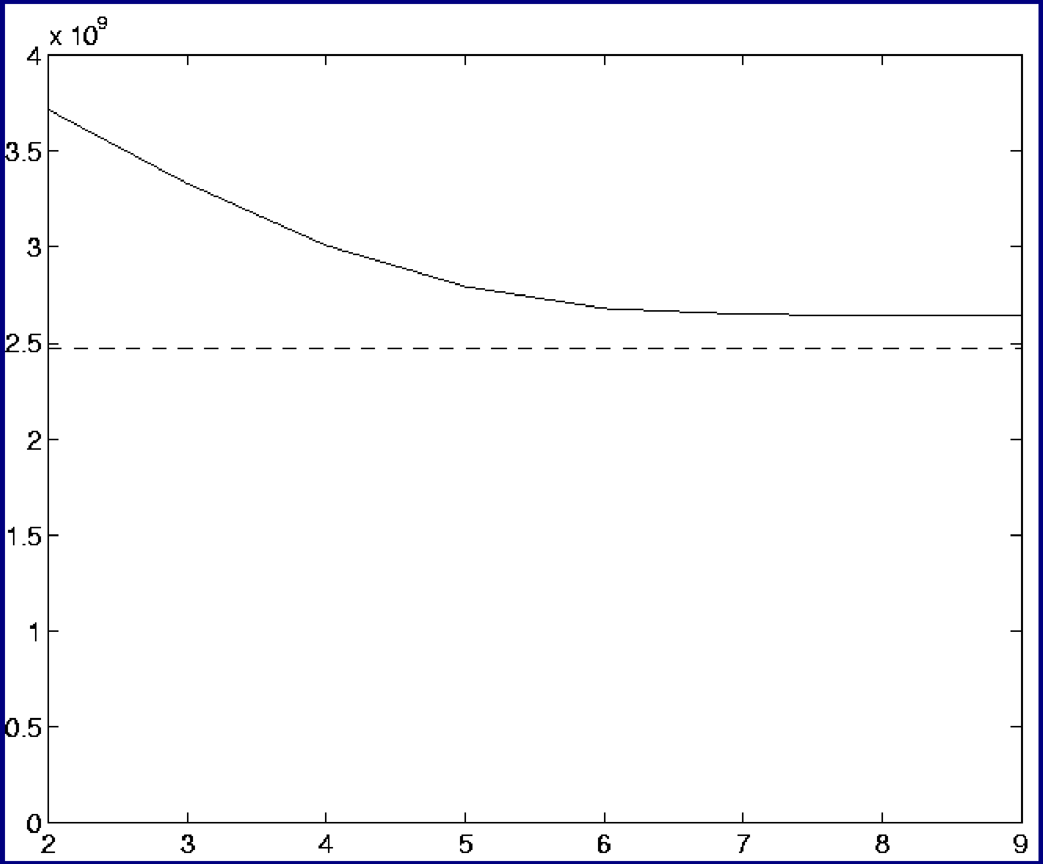
Multiresolution segmentation



Multiresolution analysis

Doesn’t the speed suffer with the additional nodes?

Computations nearly the same since the graph diameter is small.



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Conclusion

Created Graph Analysis Toolbox for MATLAB to be publicly released:

- Implementation of tools to allow processing of data associated with graphs (e.g., filtering, edge detection, Ncuts)
- Implementation of new algorithms developed in my doctoral work
- Provides tools for transferring image data from Cartesian images to graphs of varying resolution
- Provides tools for visualizing data on graphs
- Includes space-variant graph data for over 20 species
- Has scripts to generate all the figures in my dissertation and papers

Conclusion

Final message:

The analogy between graph theory, circuit theory, linear algebra and vector calculus provides

- Established principles that drive new algorithm design - with an intuition about their functionality
- An ability to transfer standard computer vision techniques to more general domains e.g., space-variant sensor data, higher dimensions, abstract feature spaces
- The potential for alternate methods of computation via circuit construction

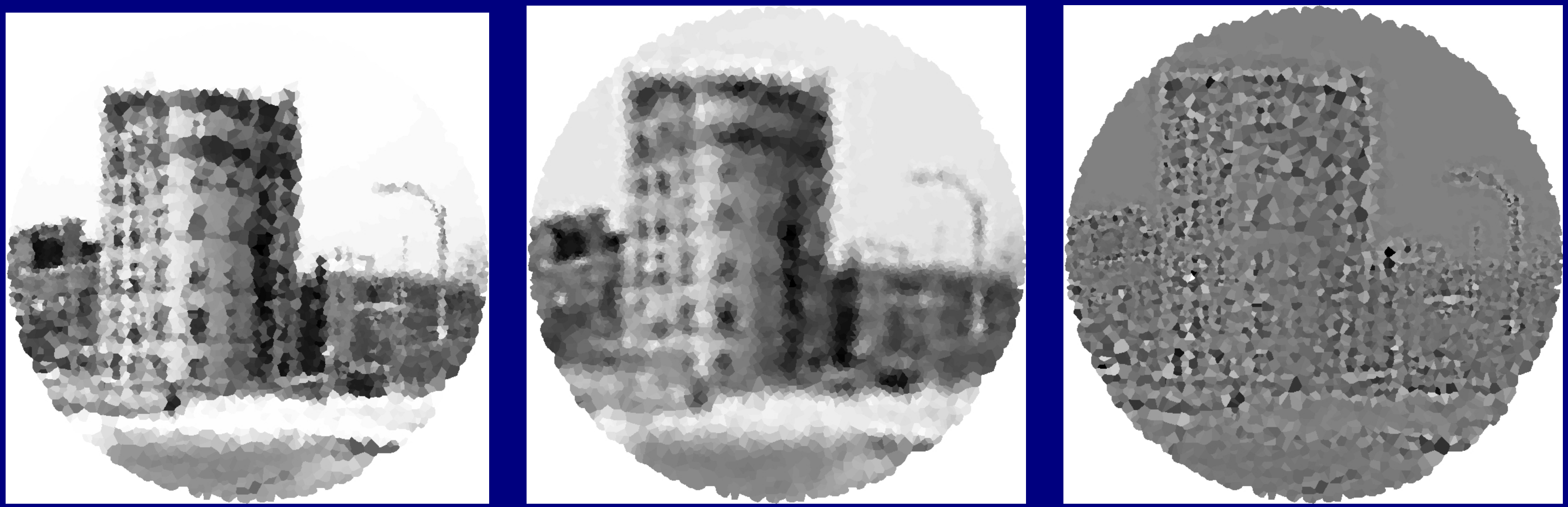
Future directions

General directions

- Use of nodes to represent image regions instead of pixels
- Incorporation of training/learning into image analysis algorithms
- Pursue interactive foreground/background segmentation
- Object recognition
- Explore relationship with statistical image analysis methods

Graph theory

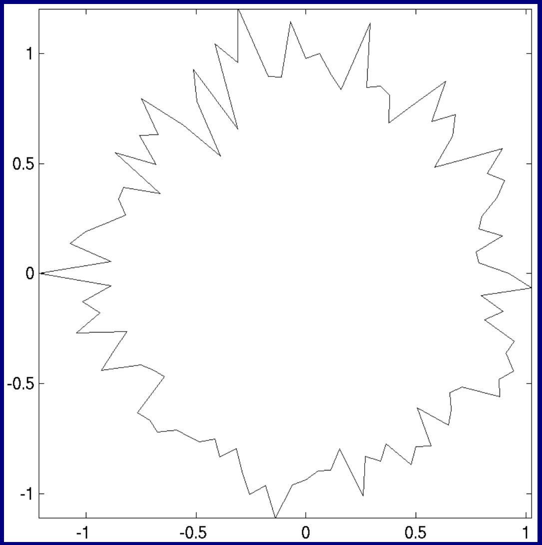
Taubin et al. (1996) frames standard filtering in the same context



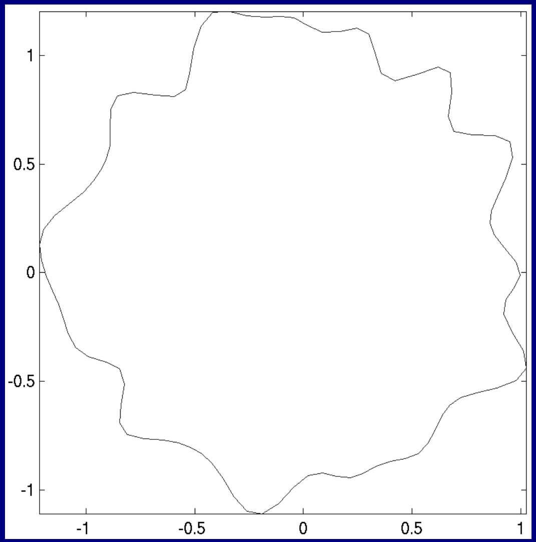
(a) Image (b) Low-pass filter (c) High-pass filter

Graph theory

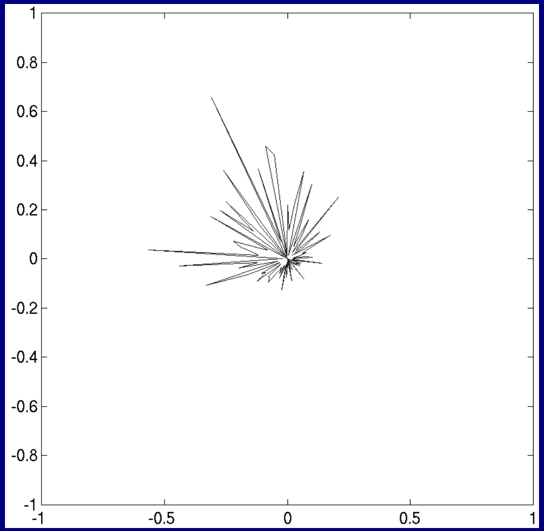
Coordinate filtering also possible (e.g., a ring graph)



(a) Original



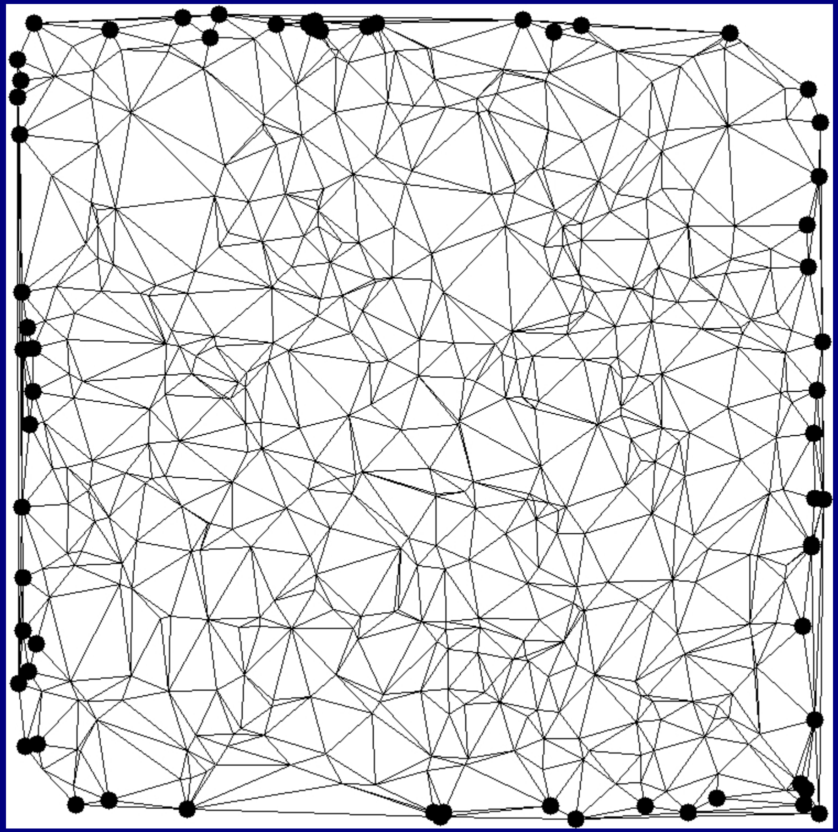
(b) Low-pass filter



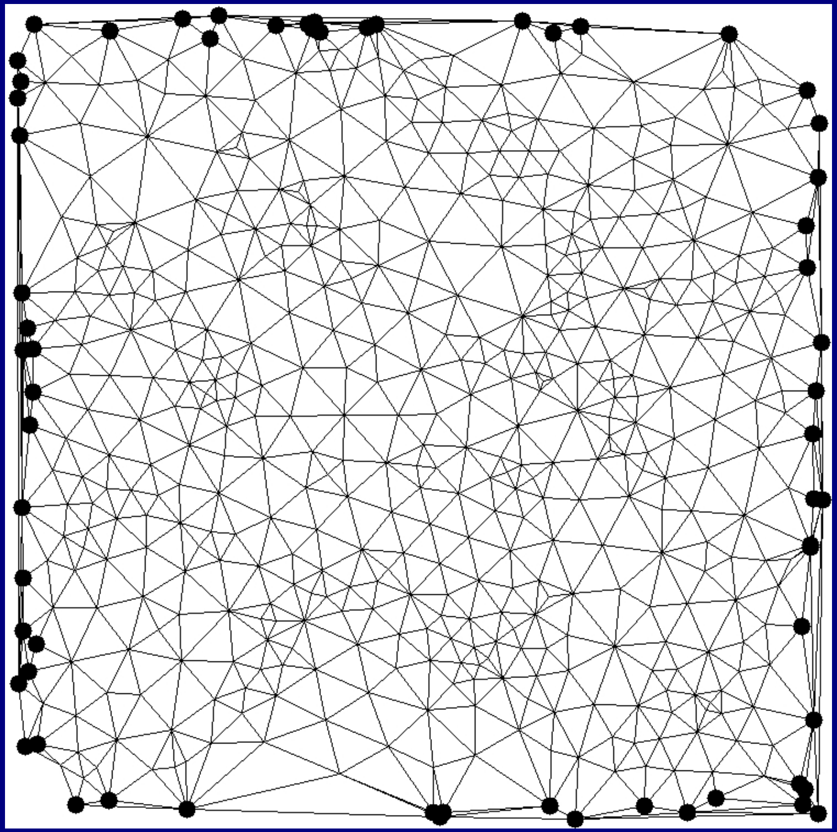
(c) High-pass filter

Dirichlet Problem

Same technique works for simple graph drawing



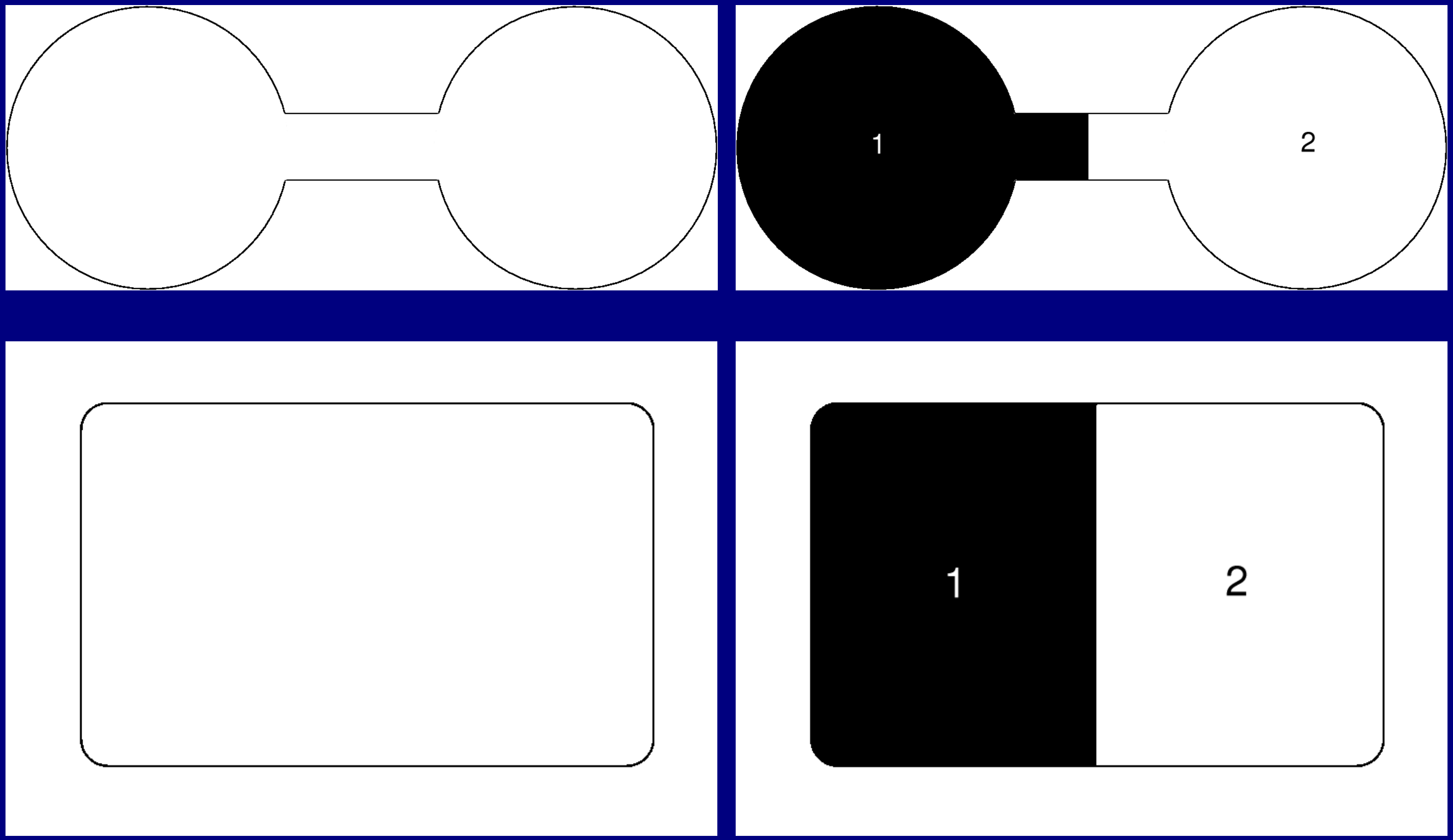
(a) Original



(b) Interpolated

Isoperimetric Problem

The isoperimetric constant quantifies the separability of a space



Spectral partitioning

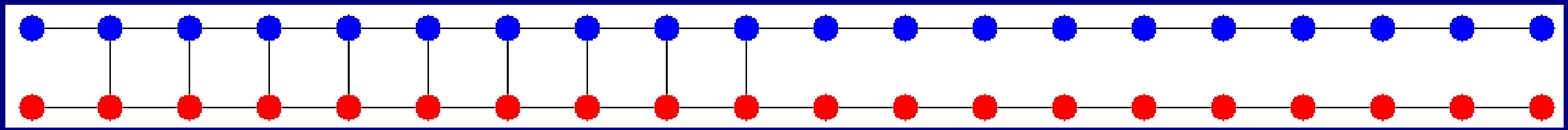
Spectral partitioning (Donath and Hoffman, 1972; Pothen et al., 1990) adopts a different approach to minimizing the isoperimetric ratio.

Isoperimetric ratio	Rayleigh quotient
$h(x) = \frac{x^T L x}{x^T d}$	$R(x) = \frac{x^T L x}{x^T x}$

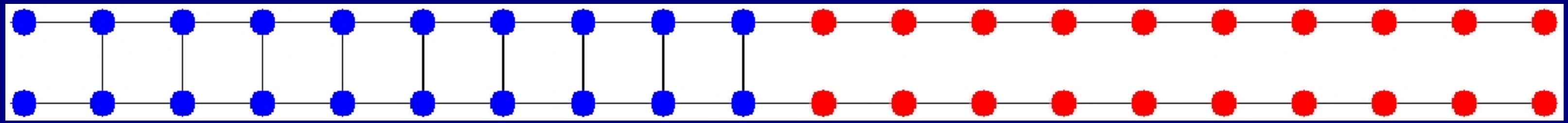
Results by Fiedler (1975a,b, 1973), Alon (1986) and Cheeger (1970) characterize the relationship between isoperimetric constant of a graph and the second smallest eigenvector of L (the **Fiedler value**). The Normalized Cuts image segmentation algorithm of Shi and Malik (2000) adopts this approach with a different representation of the Laplacian matrix.

Spectral partitioning

Guattery and Miller (1998) demonstrated that spectral partitioning fails to perform well on certain families of graphs



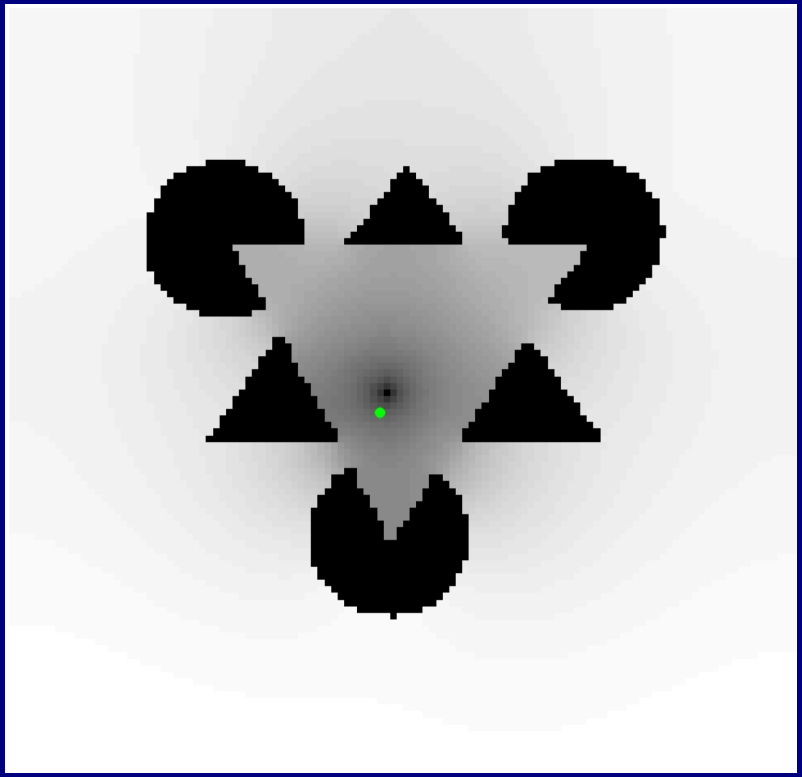
(a) Spectral



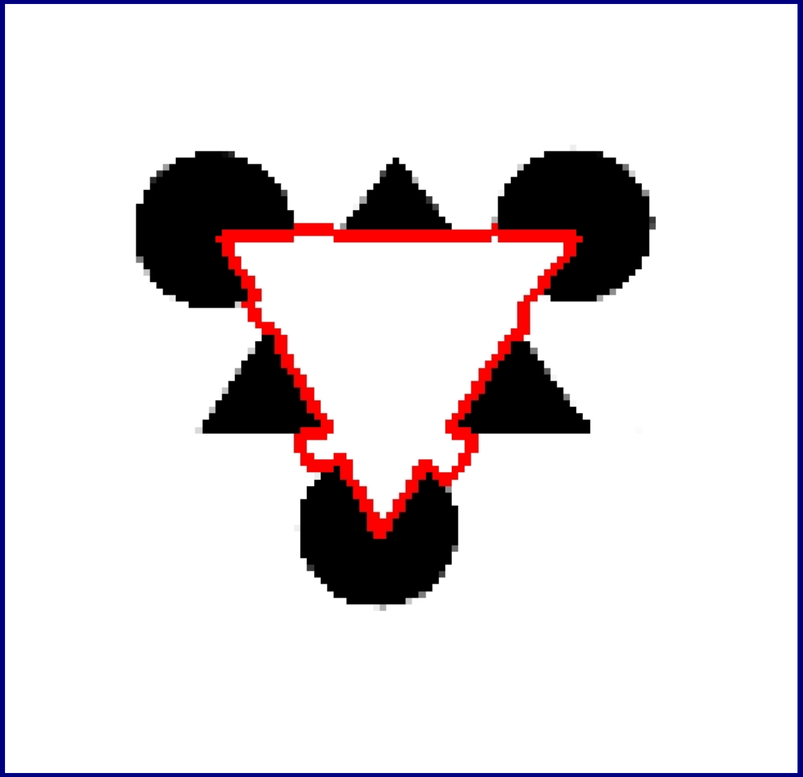
(b) Isoperimetric

Image processing

No edge information present - The Kaniza illusion



(a) Potentials



(b) Partition

References

Alon, N. (1986). Eigenvalues and expanders. *Combinatorica*, 6:83–96. 600.

Anderson, W. N. J. and Morley, T. D. (1971). Eigenvalues of the laplacian of a graph. Technical Report TR 71-45, University of Maryland.

Biggs, N. (1974). *Algebraic Graph Theory*. Number 67 in Cambridge Tracts in Mathematics. Cambridge University Press.

Branin Jr., F. H. (1966). The algebraic-topological basis for network analogies and the vector calculus. In *Generalized Networks, Proceedings*, pages 453–491, Brooklyn, N.Y.

Cheeger, J. (1970). A lower bound for the smallest eigenvalue of the laplacian. In Gunning, R., editor, *Problems in Analysis*, pages 195–199. Princeton University Press, Princeton, NJ.

Donath, W. and Hoffman, A. (1972). Algorithms for partitioning of graphs and computer logic based on eigenvectors of connection matrices. *IBM Technical Disclosure Bulletin*, 15:938–944.

Dongarra, J. J., Duff, I. S., Sorenson, D. C., and van der Vorst, H. A. (1991). *Solving Linear Systems on Vector and Shared Memory Computers*. Society for Industrial and Applied Mathematics, Philadelphia.

Fiedler, M. (1973). Algebraic connectivity of graphs. *Czechoslovak Mathematical Journal*, 23(98):298–305.

Fiedler, M. (1975a). Eigenvalues of acyclic matrices. *Czechoslovak Mathematical Journal*, 25(100):607–618.

Fiedler, M. (1975b). A property of eigenvalues of nonnegative symmetric matrices and its applications to graph theory. *Czechoslovak Mathematical Journal*, 25(100):619–633.

Guattery, S. and Miller, G. (1998). On the quality of spectral separators. *SIAM Journal on Matrix Analysis and Applications*, 19(3):701–719.

Hughes, A. (1977). The topography of vision in mammals of contrasting life style: Comparative optics and retinal organization. In *The Handbook of Sensory Physiology*, volume 7. New York: Academic Press.

Perona, P. and Malik, J. (1990). Scale-space and edge detection using anisotropic diffusion. *IEEE Transac-*

tions on Pattern Analysis and Machine Intelligence, 12(7):629–639.

Pothen, A., Simon, H., and Liou, K.-P. (1990). Partitioning sparse matrices with eigenvectors of graphs. *SIAM Journal of Matrix Analysis Applications*, 11(3):430–452.

Shi, J. and Malik, J. (2000). Normalized cuts and image segmentation. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 22(8):888–905.

Taubin, G., Zhang, T., and Golub, G. (1996). Optimal surface smoothing as filter design. Technical Report RC-20404, IBM.