Faster Graph-Theoretic Image Processing via Small-World and Quadtree Topologies

Leo Grady
Siemens Corporate Research
(formerly Boston University, Department of Cognitive and Neural Systems)

Eric Schwartz
Boston University, Depts. of Cognitive and Neural Systems,
Electrical and Computer Engineering

Talk Outline

1. Introduction – Graphs, images and topology
2. Designing a topology
3. Small world graphs
4. Multiresolution graphs
5. Conclusion

Graphs, Images and Topology

Zahn, C., “Graph Theoretical Methods for Detecting and Describing Gestalt Clusters”, IEEE Trans. on Computation, (20) 1971

Shi, J. and Malik, J. “Normalized Cuts and Image Segmentation”, IEEE PAMI, 8, (22) 2000

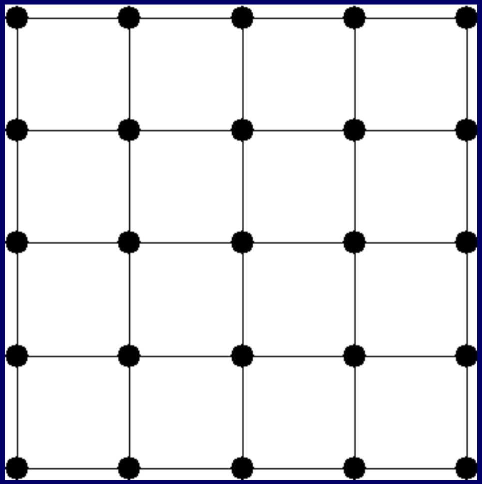
A **graph** $G=(V,E)$ with vertices $v \in V$, edges $e \in E \subseteq V \times V$.

Topological space: $\{X,\tau\}$, where X, represents a set of elements and τ, defines a collection of subsets of X.

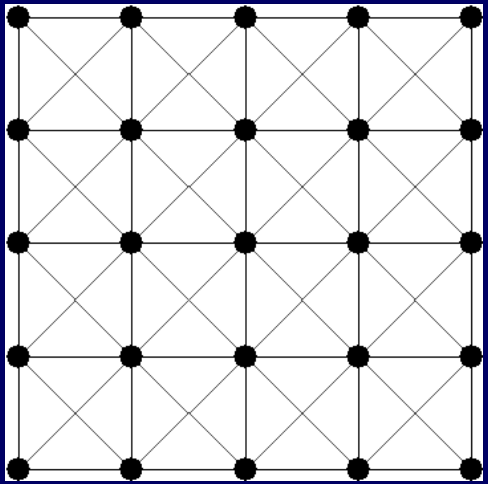
Graphs, Images and Topology

Standard image topologies

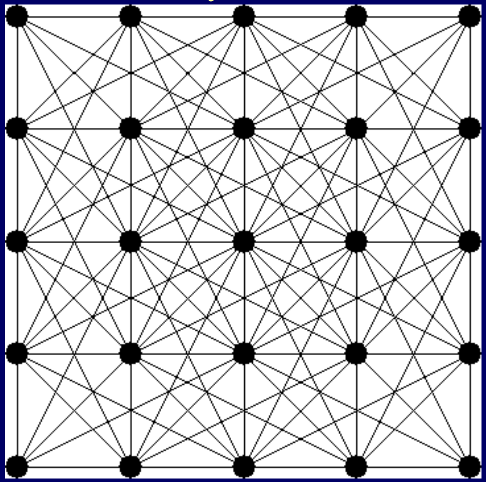
4-connected



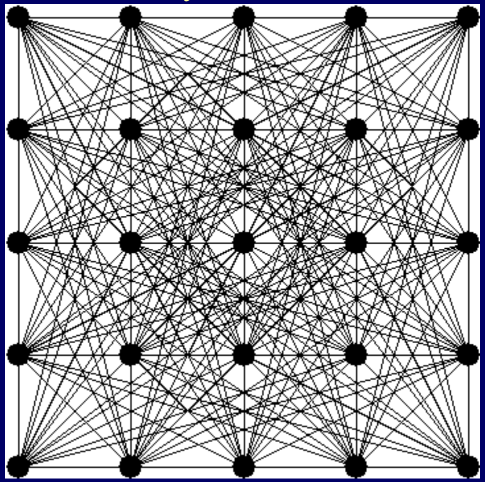
8-connected



Radially connected



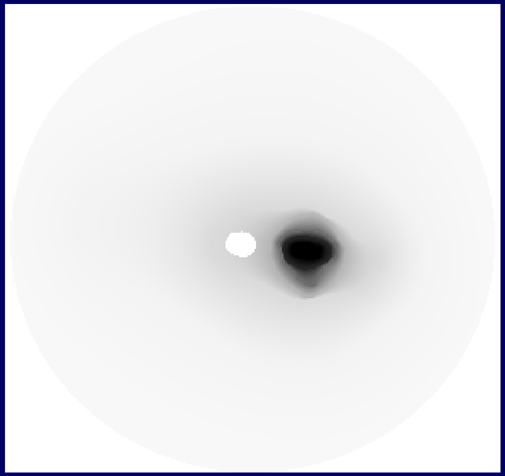
Fully connected



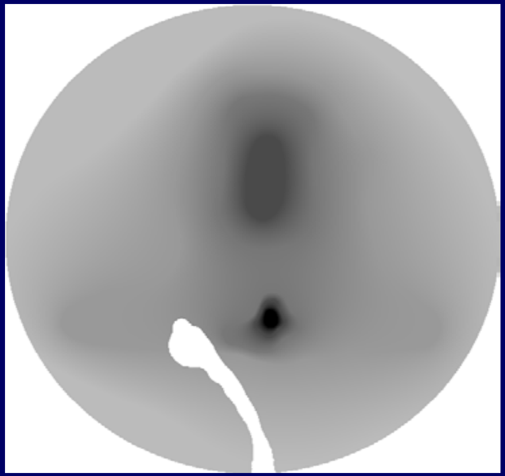
Graphs, Images and Topology

Photoreceptor densities:

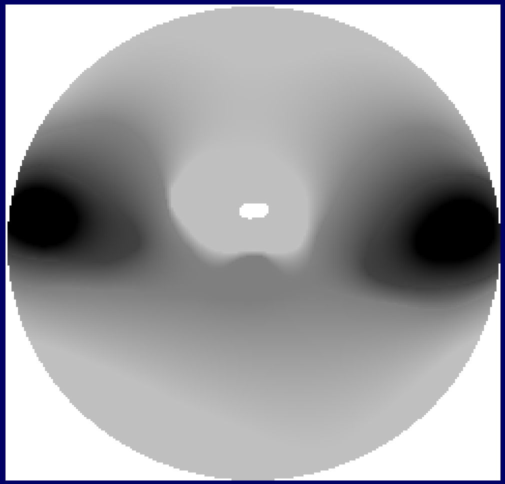
Baboon



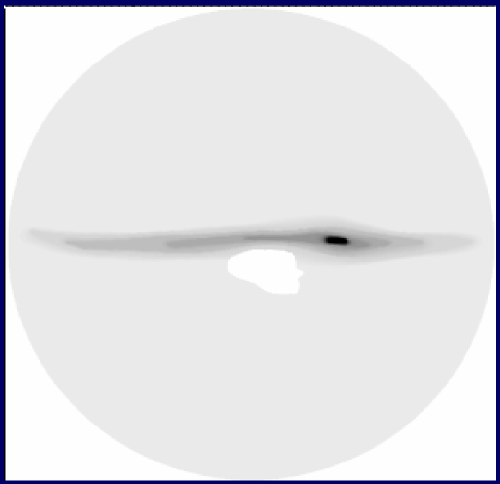
Pigeon



Dolphin



Cheetah



Evolution has driven irregular representations

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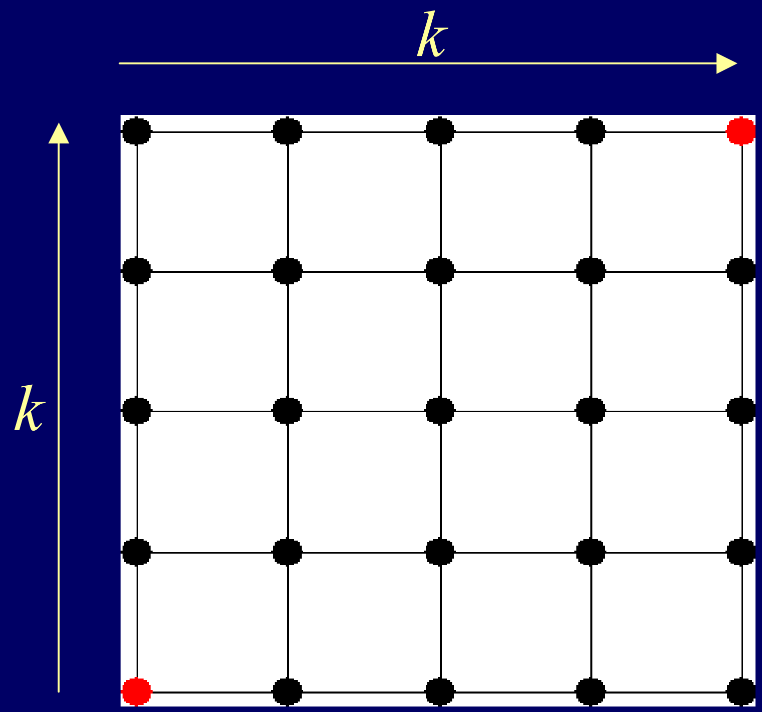
Designing a Topology

Principle: Choose a topology that increases algorithmic efficiency while maintaining result quality.

Graph **diameter:**



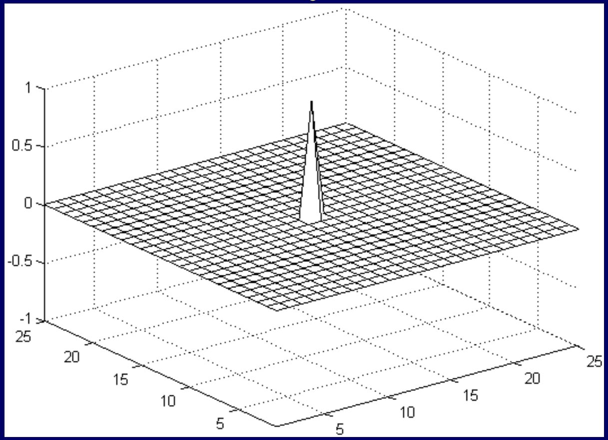
Diameter $= 2k$

Iterative matrix methods may be thought of as a *mixing process*

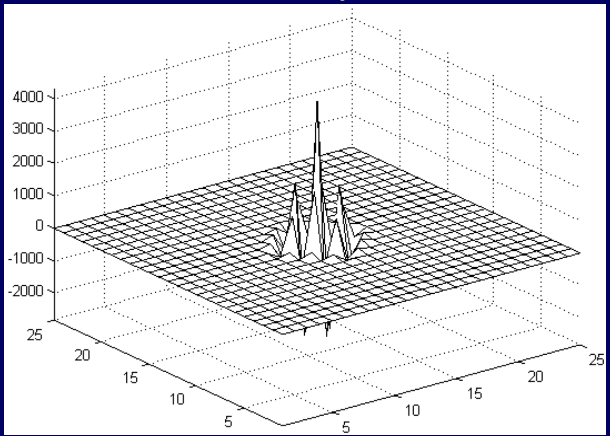
Designing a Topology

Example: Krylov subspace bases at progressive iterations of CG with graph (finite difference) Laplacian matrix

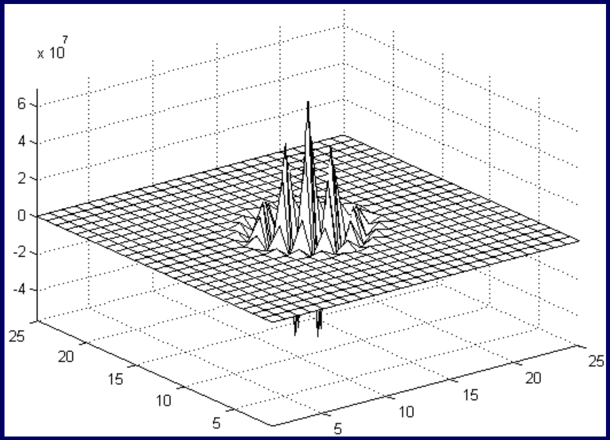
r_0



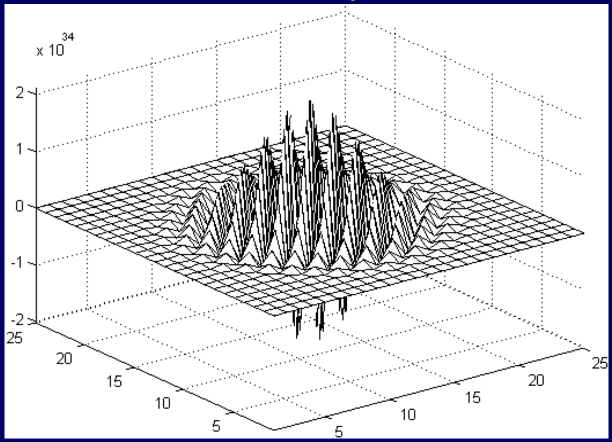
$L^5 r_0$



$L^{10} r_0$



$L^{40} r_0$

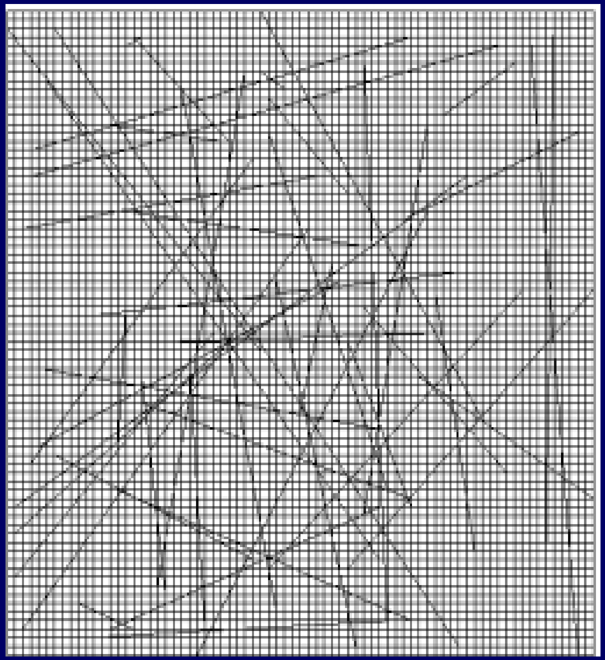


Designing a Topology

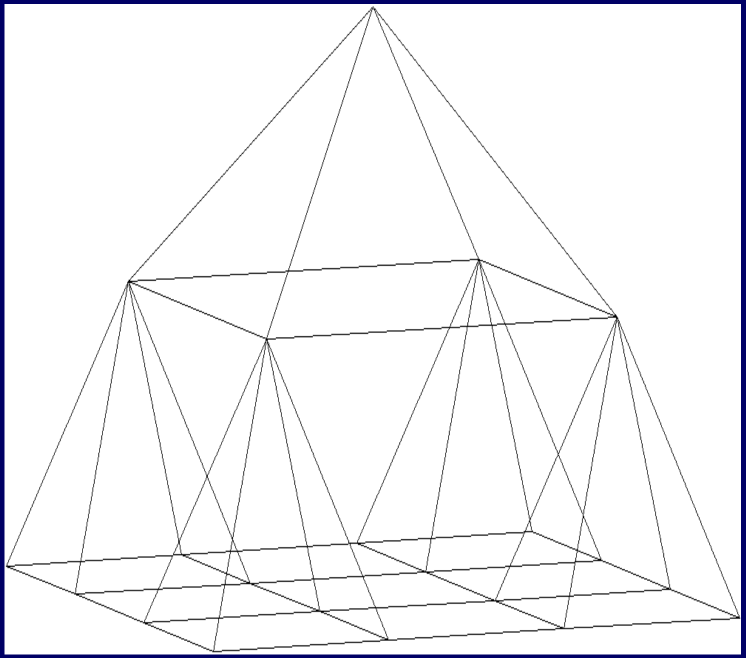
Various image processing techniques use iterative matrix methods to perform the required computations, including elliptic PDEs, diffusion, isoperimetric segmentation algorithm.

We propose two principled topologies that give small graph diameter:

Small world graph



Connected pyramid



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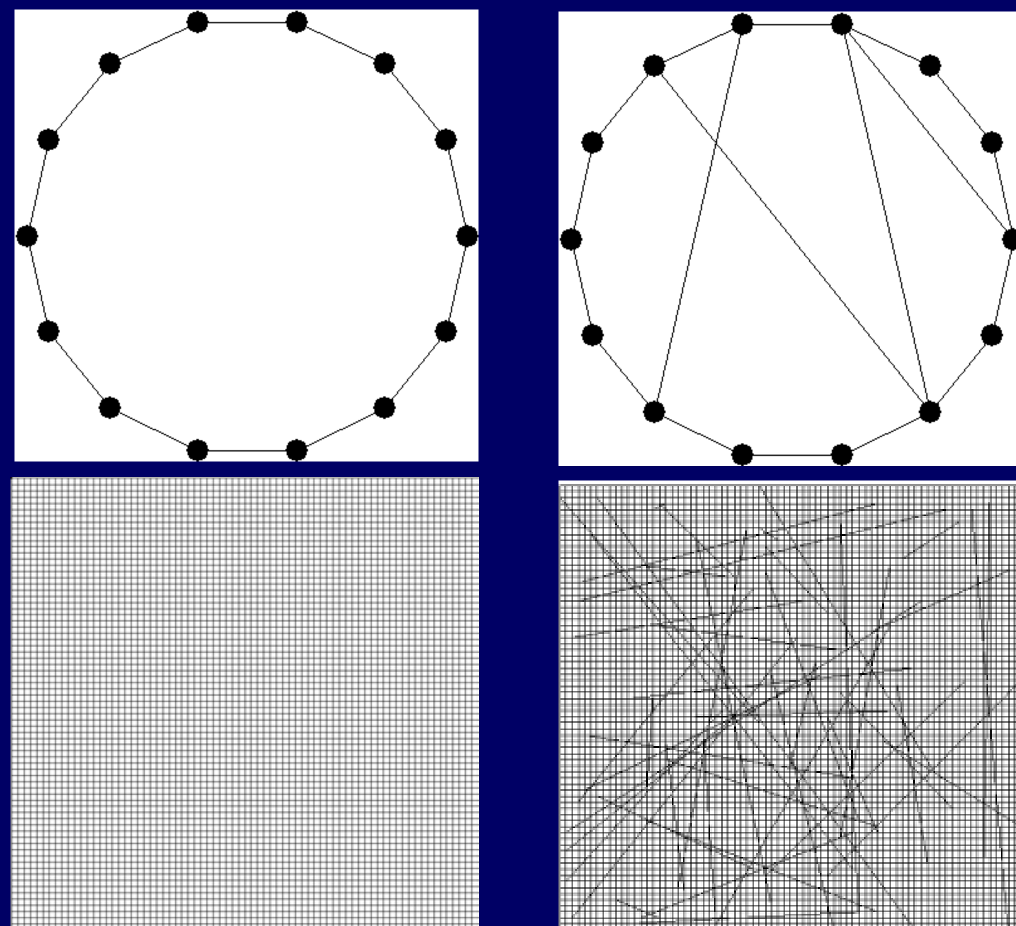
Small world graphs

A **small-world** graph is defined as a graph that is primarily locally connected but has small diameter. A locally connected **substrate graph** may be transformed into a small world graph via the introduction of random edges.

Substrate

Small world



Diameter = 150

Diameter = 46

Small world graphs

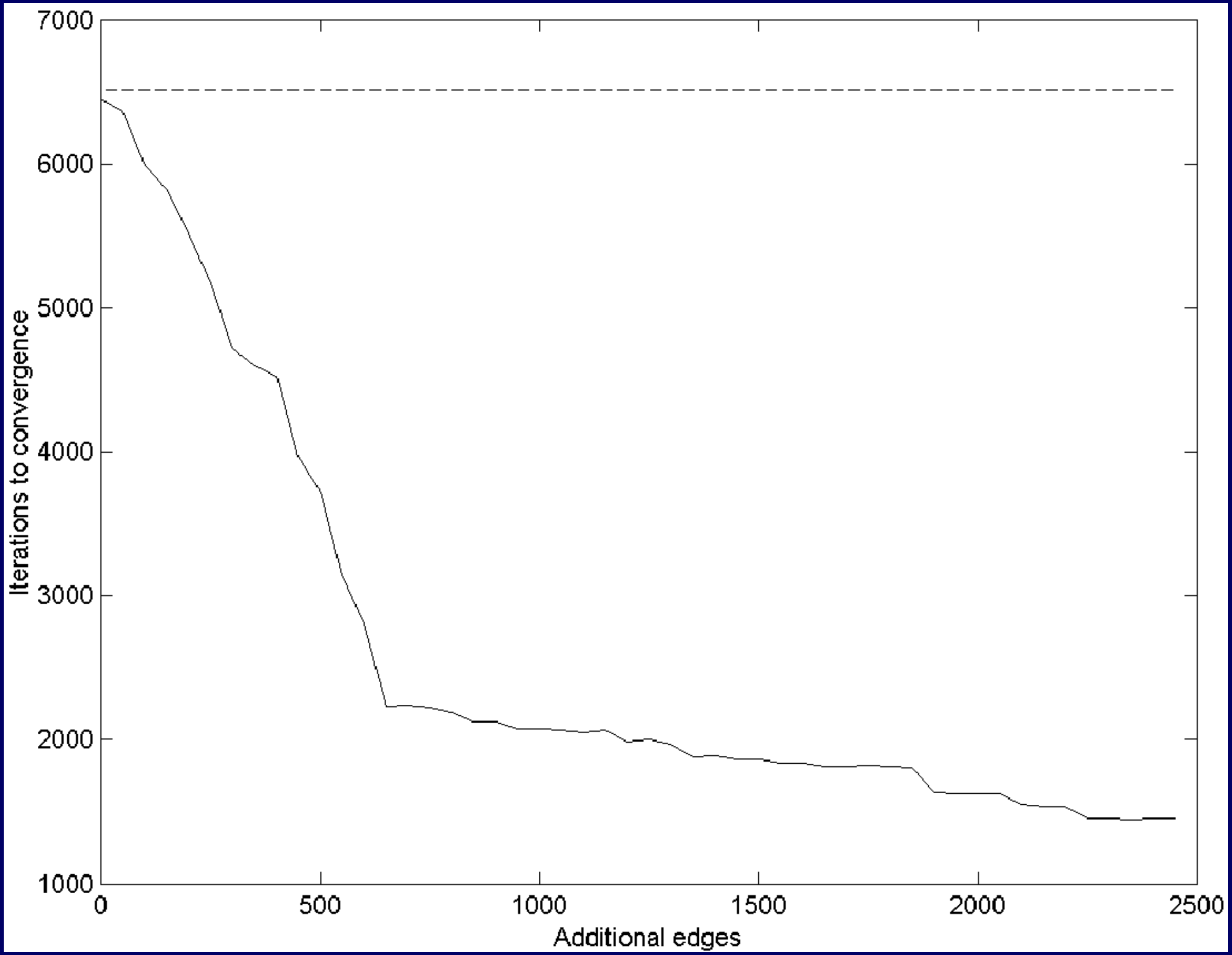
Proposal: Add a small number of random edges to a substrate graph (e.g., 4-connected lattice) in order to produce a small world graph, which retains a primarily local connectivity but has a dramatically smaller diameter.

Two issues to address:

1. At what rate does the convergence of an iterative method increase?
2. Will the results be adversely affected by the addition of random edges?

Small world graphs

Convergence of CG as additional edges added:

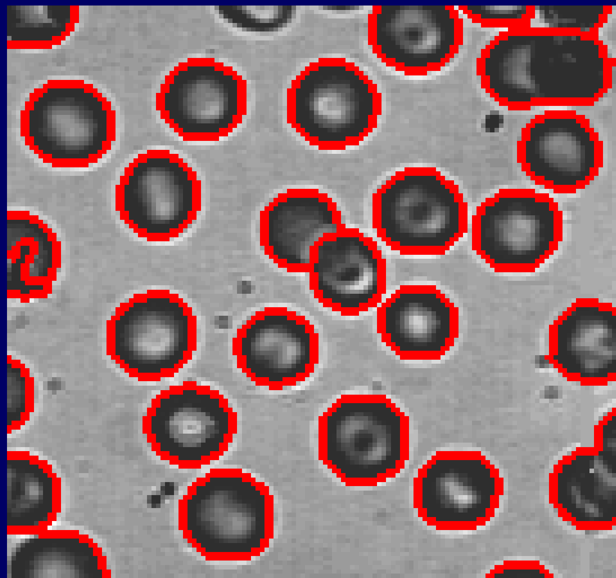


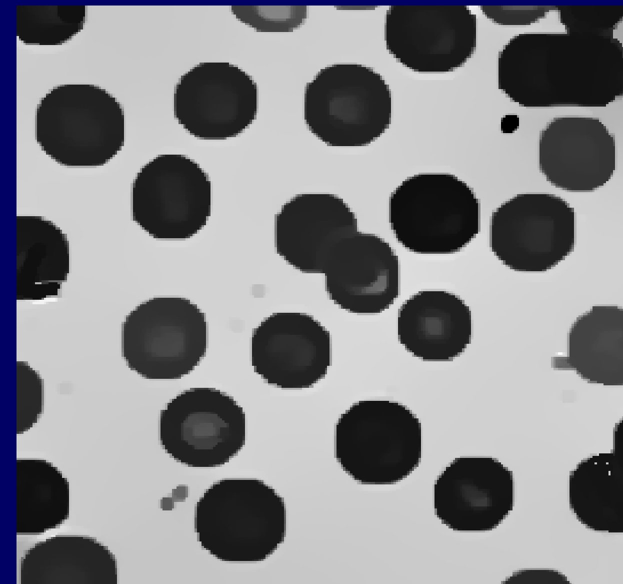
Small world graphs

Isoperimetric segmentation

Anisotropic diffusion

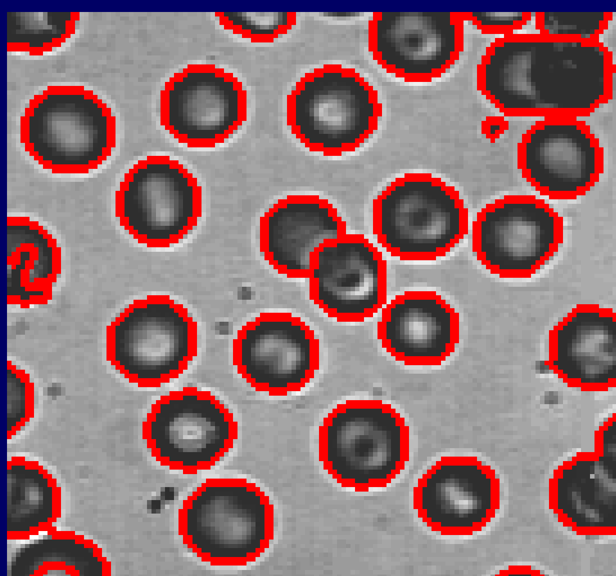
4-Connected Lattice

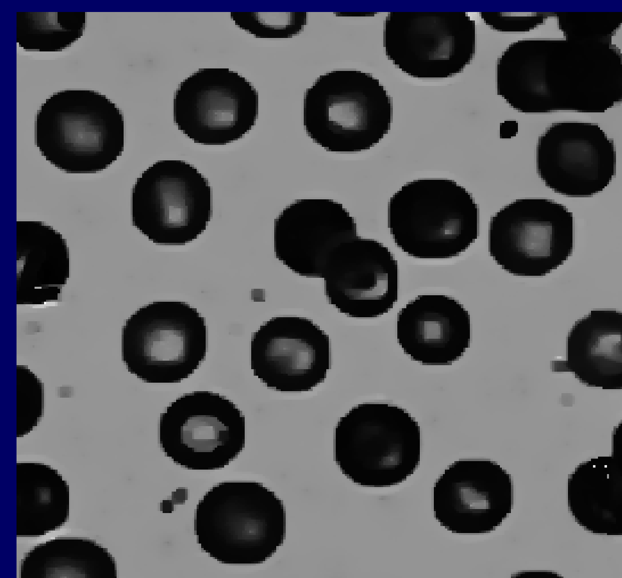




Iterations 1209

Small world





Iterations 475

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Multiresolution graphs

Common approach:

Image

↓

Filter

↓

Downsample

↓

Process

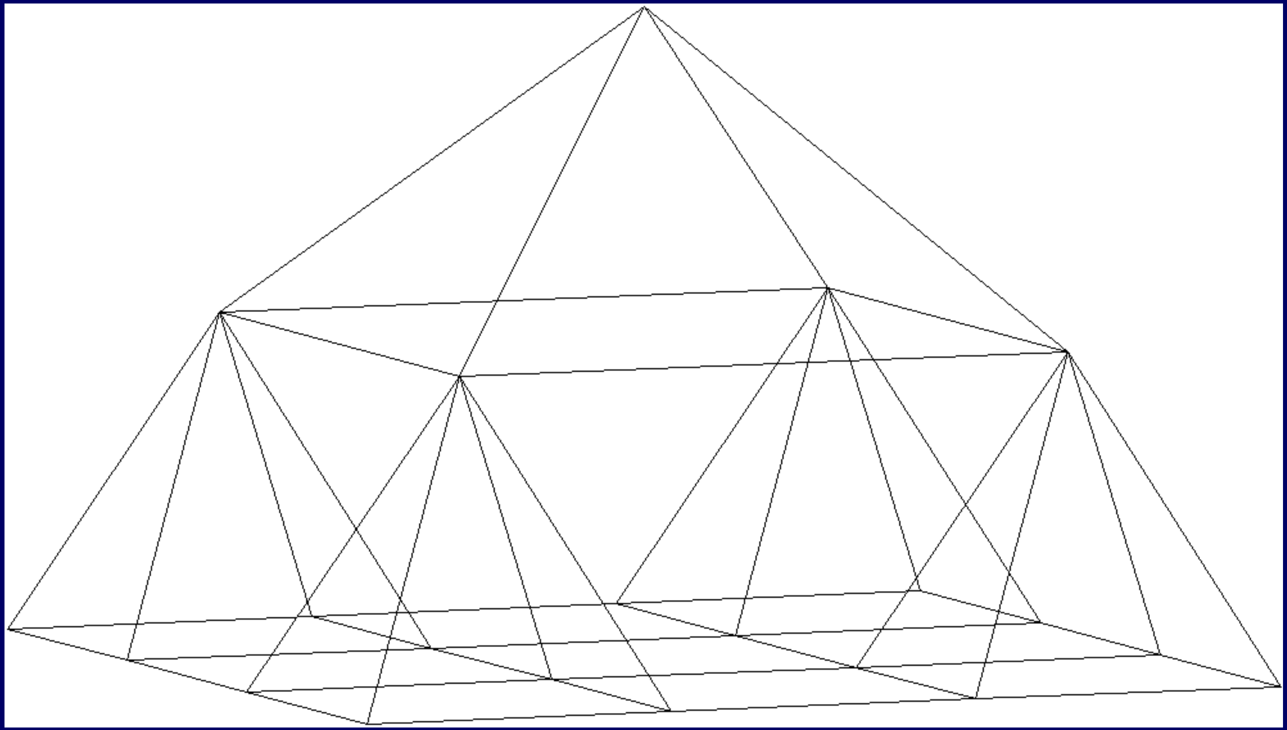
↓

Backproject solution

Used for speed and robustness to noise

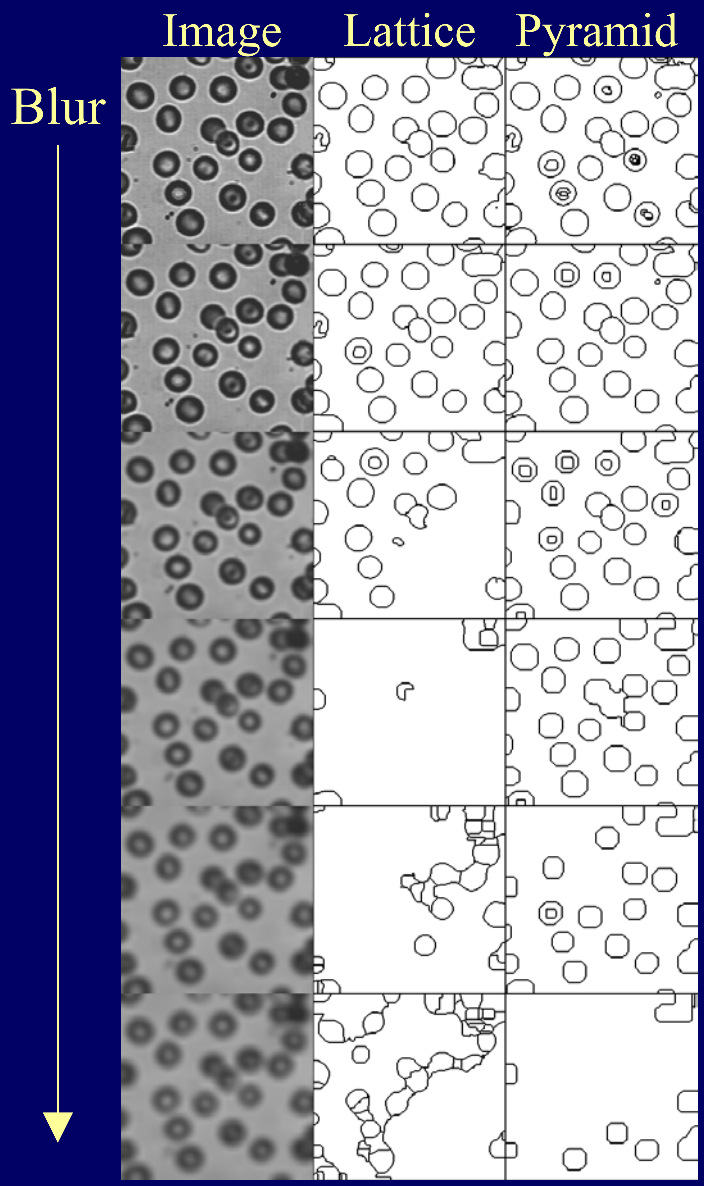
Multiresolution graphs

Idea: Apply a graph-based analysis algorithm to the whole pyramid as a *separate graph*.



Multiresolution graphs

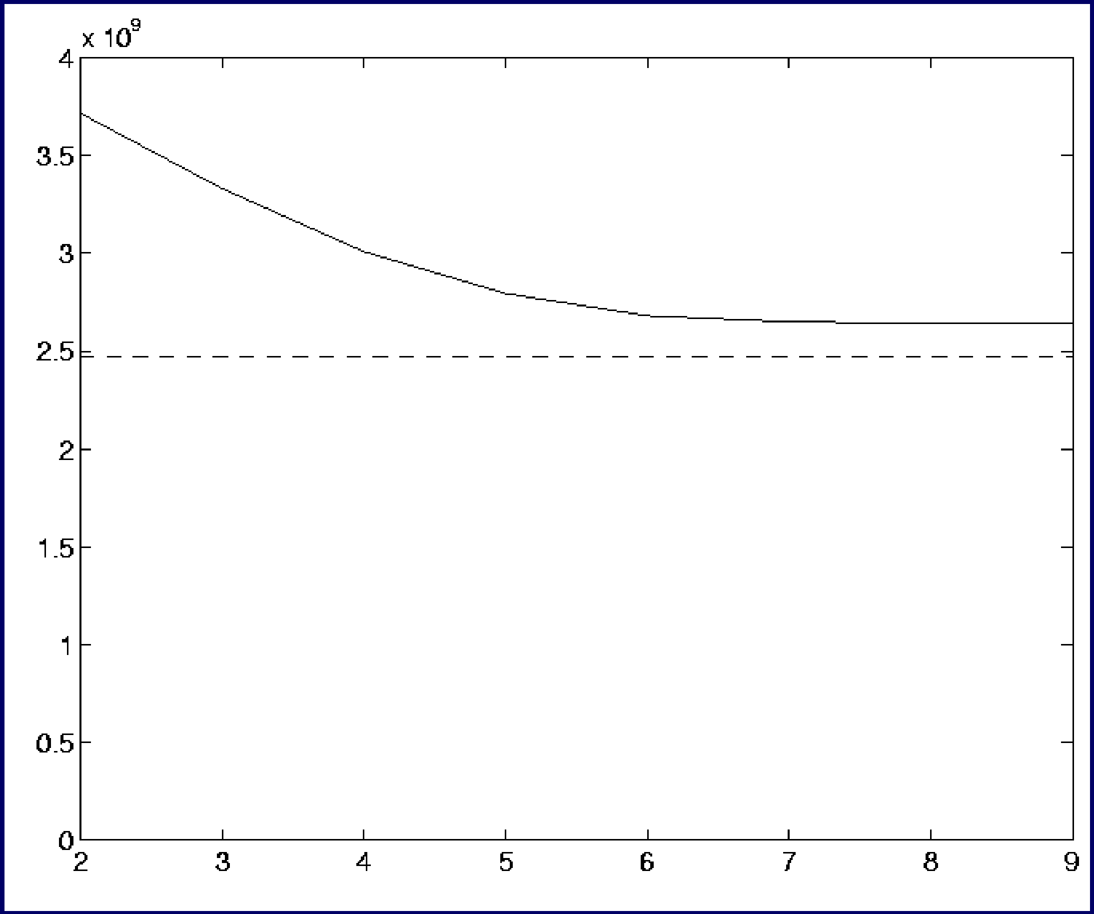
Long range connections improve performance on blurry images



Multiresolution graphs

Doesn't speed suffer from additional nodes?

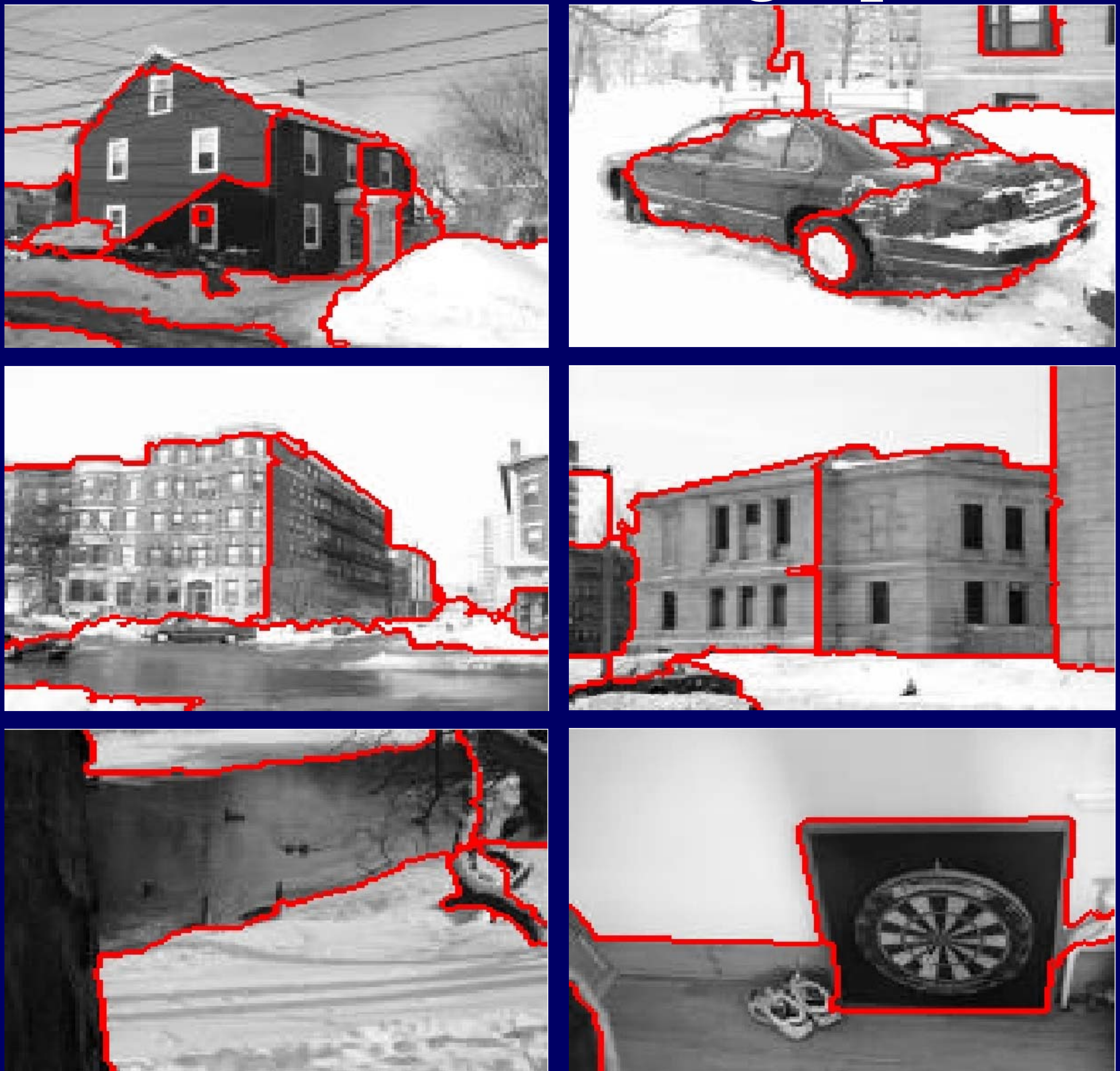
Computations nearly the same, due to small diameter



Computations

Number of levels

Multiresolution graphs



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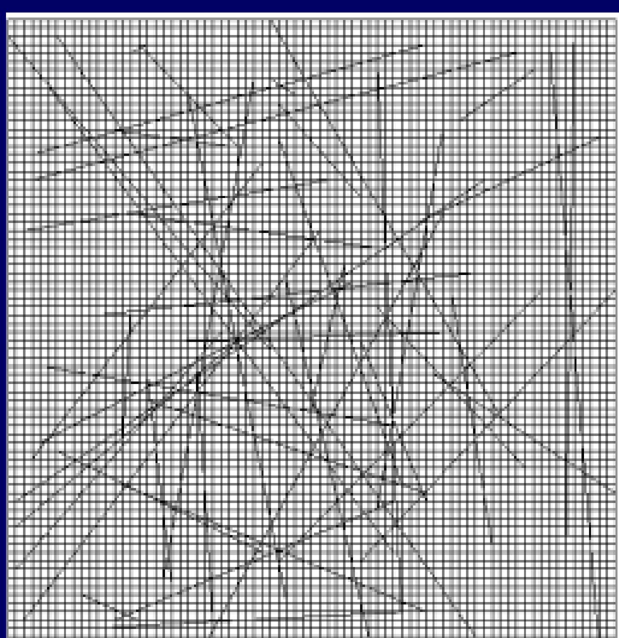
Conclusion

We asked: Can we design an image topology that enhances algorithm performance?

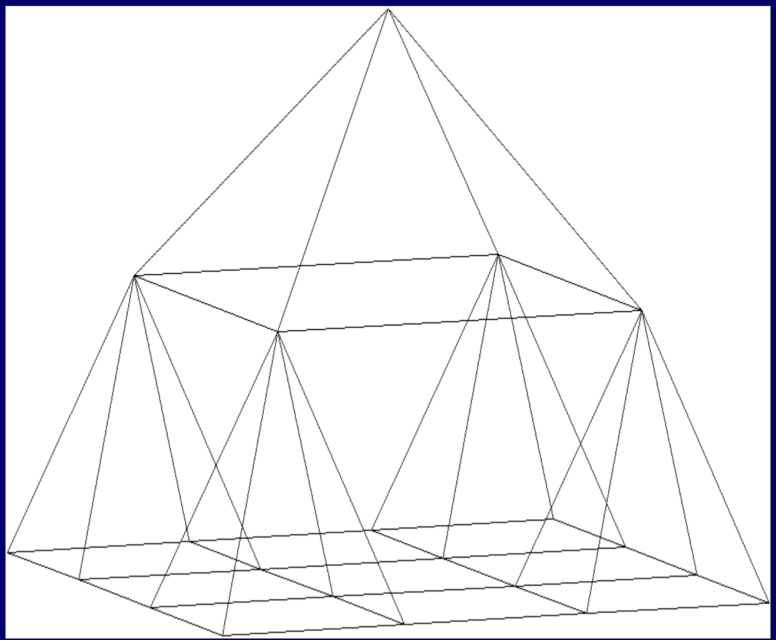
Exploit rapid convergence of mixing algorithms on graphs of low diameter.

We proposed:

Small world graph



Connected pyramid



Conclusion

Reproducible research

Images used in paper copyright free at:
http://eslab.bu.edu/resources/imageDB/imageDB.php

MATLAB toolbox for graph theoretic image processing at:
http://eslab.bu.edu/software/graphanalysis/

MATLAB scripts that generate each figure in paper at:
http://eslab.bu.edu/publications/publications.php#grady2004faster

Isoperimetric algorithm technical report:
http://eslab.bu.edu/publications/publications.php#grady2003isoperimetric-TR
My PhD thesis:
http://eslab.bu.edu/publications/publications.php#grady2004space_phd

More image segmentations computed with the pyramid-based algorithm at:
http://eslab.bu.edu/publications/grady2004faster/

Isoperimetric algorithm

Idea: Solve combinatorial isoperimetric problem by finding a partition that minimizes the ratio

$$h = \inf_S \frac{|\partial S|}{\mathrm{Vol}_S}$$

Indicator function defined on the nodes relaxed to real value.
Minimization of ratio achieved by regularization through "grounding" of an arbitrary node.

$$L_M x = d$$